The Natural Logarithmic Function

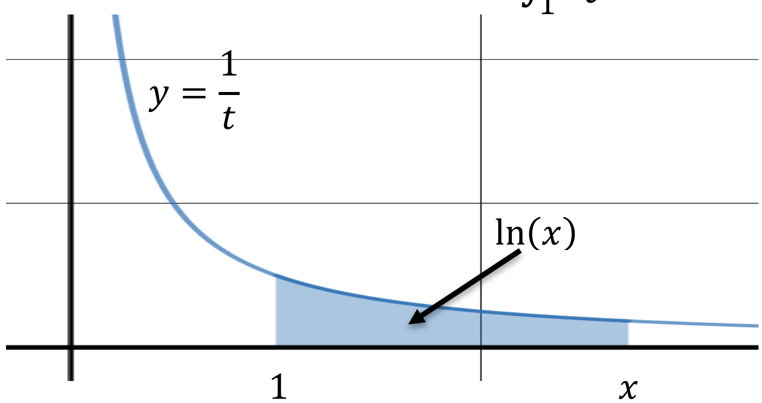
When studying algebra one often sees log to the base b $(b > 0, b \neq 1)$ defined by saying:

$$y = b^x \text{ if, and only if, } x = \log_b y$$

One problem with this approach is that it was not clear what was meant by $2^{\sqrt{2}}$.

Def. The **natural logarithm** of a number $x > 0$ is given as:

$$\ln x = \int_1^x \frac{1}{t}\,dt$$



This definition makes sense for any $x > 0$. Notice the following:

1) If $x > 1$, then:

$$\ln(x) = \int_1^x \frac{1}{t}\,dt > 0$$

2) If $0 < x < 1$, then:

$$\ln(x) = \int_1^x \frac{1}{t}\,dt < 0$$

For example

$$\ln\left(\frac{1}{2}\right) = \int_1^{\frac{1}{2}} \frac{1}{t}\,dt = -\int_{\frac{1}{2}}^1 \frac{1}{t}\,dt < 0$$

3) If $x = 1$, then:

$$\ln(1) = \int_1^1 \frac{1}{t}\, dt = 0$$

4) By The Fundamental Theorem of Calculus:

$$\frac{d}{dx}(\ln(x)) = \frac{d}{dx}\left(\int_1^x \frac{1}{t}\, dt\right) = \frac{1}{x}$$

5) Since $\frac{d}{dx}(\ln x) = \frac{1}{x}$ exists for all $x > 0$, then we can say $y = \ln x$ is continuous for all $x > 0$.

6) Since $\frac{d}{dx}(\ln x) = \frac{1}{x} > 0$ for $x > 0$, then we can say $y = \ln x$ is an increasing function for $x > 0$.

7) $\frac{d^2}{dx^2}(\ln x) = \frac{d}{dx}\left(\frac{1}{x}\right) = -\frac{1}{x^2} < 0$ for $x > 0$ so the graph of $y = \ln x$ is concave down for $x > 0$.

Logarithm laws for $x, y > 0$ and r, a rational number:

1) $\ln(xy) = \ln x + \ln y$

2) $\ln\left(\frac{x}{y}\right) = \ln x - \ln y$

3) $\ln(x^r) = r \ln x$

Proof of #1: $\ln(xy) = \ln x + \ln y$

Let $f(x) = \ln(ax)$, where $a > 0$, then by the chain rule we can write:

$$f'(x) = \left(\frac{1}{ax}\right)\frac{d}{dx}(ax) = \frac{1}{ax}\ (a) = \frac{1}{x}.$$

So $f(x) = \ln ax$ and $g(x) = \ln x$ have the same derivative, thus:

$$\ln(ax) = \ln x + C.$$

Now let $x = 1$:

$$\ln(a) = \ln 1 + C = C$$

$$\Longrightarrow \qquad \ln(ax) = \ln x + \ln a.$$

Replacing "a" with y, we get:

$$\ln(xy) = \ln x + \ln y.$$

Ex. Using the logarithm laws expand $\ln\left(\frac{(x^4+2)^5 \cos x}{\sqrt{x^2+1}}\right)$.

$$\ln\left(\frac{(x^4+2)^5 \cos x}{\sqrt{x^2+1}}\right) = \ln\left(\frac{(x^4+2)^5 \cos x}{(x^2+1)^{\frac{1}{2}}}\right)$$

$$= \ln((x^4+2)^5 \cos x) - \ln(x^2+1)^{\frac{1}{2}} \quad ; \qquad \ln\left(\frac{x}{y}\right) = \ln x - \ln y$$

$$= \ln((x^4+2)^5) + \ln \cos x - \ln(x^2+1)^{\frac{1}{2}} \quad ; \quad \ln(xy) = \ln x + \ln y$$

$$= 5\ln(x^4+2) + \ln \cos x - \frac{1}{2}\ln(x^2+1) \quad ; \qquad \ln(x^r) = r\ln x.$$

Using the logarithm laws, we can see:

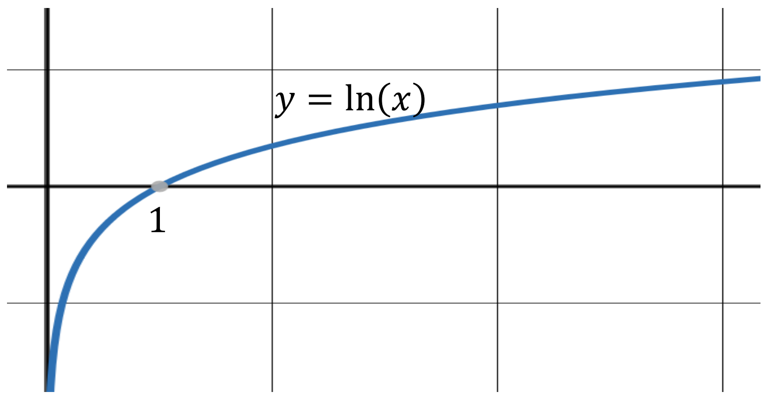
$$\lim_{x\to\infty} \ln x = +\infty \quad \text{and} \quad \lim_{x\to 0^+} \ln x = -\infty \ \text{ since:}$$

$$\ln 2 > 0 \Longrightarrow \lim_{n\to\infty} \ln(2^n) = \lim_{n\to\infty} n \ln 2 = +\infty \quad ;$$

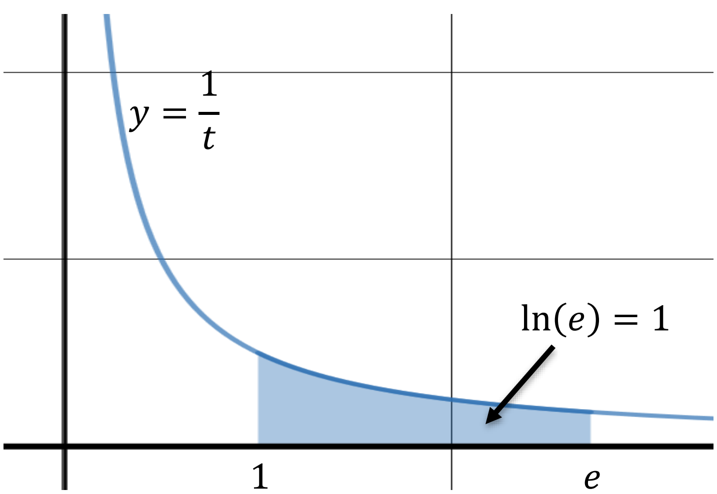
If $t = \frac{1}{x}$, then:

$$\lim_{x\to 0^+} \ln x = \lim_{t\to\infty} \ln\left(\frac{1}{t}\right) = \lim_{t\to\infty}(-\ln t) = -\infty.$$

Now we can sketch a graph of $y = \ln x$ where the y-axis, $x = 0$, is a vertical asymptote.



Def. $\boldsymbol{e}$ is the number such that $\ln e = 1$.



$$\ln e = \int_1^e \frac{1}{t}\, dt = 1$$

$$e \approx 2.718.$$

So far we have seen that if $y = \ln x$, then $\dfrac{dy}{dx} = \dfrac{1}{x}$. If we apply the chain rule, then:

$$\boldsymbol{\frac{d}{dx}\ln\big(u(x)\big) = \frac{1}{u(x)}\frac{du}{dx}} .$$

Ex. Find $\dfrac{dy}{dx}$ for the following functions:

a) $y = x^2 \ln x$
b) $y = 3\ln(2 + \sin x)$

a) $y = x^2 \ln x$

$$\frac{dy}{dx} = x^2\left(\frac{d}{dx}(\ln x)\right) + (\ln(x))\left(\frac{d}{dx}(x^2)\right)$$

$$= x^2\left(\frac{1}{x}\right) + (\ln x)(2x) = x + 2x\ln x .$$

b) $y = 3\ln(2 + \sin x)$

$$\frac{dy}{dx} = 3\left(\frac{1}{2+\sin x}\right)\frac{d}{dx}(2 + \sin x)$$

$$= \frac{3}{2+\sin x}(\cos x) = \frac{3\cos x}{2+\sin x} .$$

Ex. Let $y = \ln\left(\dfrac{(x^4+2)^5 \cos x}{\sqrt{x^2+1}}\right)$. Find $\dfrac{dy}{dx}$. From an earlier example we know:

$$y = 5\ln(x^4 + 2) + \ln(\cos x) - \frac{1}{2}\ln(x^2 + 1)$$

$$\frac{dy}{dx} = 5\left(\frac{1}{x^4+2}\right)\frac{d}{dx}(x^4 + 2) + \frac{1}{\cos x}\frac{d}{dx}(\cos x) - \frac{1}{2}\left(\frac{1}{x^2+1}\right)\frac{d}{dx}(x^2 + 1)$$

$$= \frac{5}{x^4+2}(4x^3) + \frac{1}{\cos x}(-\sin x) - \frac{1}{2}\left(\frac{1}{x^2+1}\right)(2x)$$

$$= \frac{20x^3}{x^4+2} - \tan x - \frac{x}{x^2+1} .$$

One could use the chain rule directly on the previous function but that would be a very messy calculation.

Ex. Find $\frac{dy}{dx}$ for the following:

a) $y = \ln(\ln(x^2))$
b) $y = \cos(\ln x) + \ln(\sin x)$

a) $y = \ln(\ln(x^2)) = \ln(2\ln x)$

$$\frac{dy}{dx} = \frac{1}{2\ln x}\frac{d}{dx}\big((2\ln x)\big) = \frac{1}{2\ln x}\left(\frac{2}{x}\right) = \frac{1}{x\ln x}$$

b) $y = \cos(\ln x) + \ln(\sin x)$

$$\frac{dy}{dx} = -(\sin(\ln x))\frac{d}{dx}(\ln x) + \frac{1}{\sin x}\frac{d}{dx}(\sin x)$$

$$= -(\sin(\ln x))\frac{1}{x} + \frac{1}{\sin x}(\cos x)$$

$$= -\frac{\sin(\ln x)}{x} + \cot x$$

Ex. Find $\frac{d}{dx}(\ln|x|)$. Notice that $f(x) = \ln|x|$ is defined for all real numbers where $x \neq 0$, since $|x| > 0$ if $x \neq 0$.

If $x > 0$, then $|x| = x$ and $\ln|x| = \ln x$. So $\frac{d}{dx}(\ln|x|) = \frac{1}{x}$.

If $x < 0$, then $|x| = -x$. Let $u = -x$ and take:

$$\frac{d}{dx}(\ln u) = \frac{d(\ln u)}{du}\frac{du}{dx} = \frac{1}{u}(-1) = \frac{1}{-x}(-1) = \frac{1}{x}.$$

So $\frac{d}{dx}(\ln|x|) = \frac{1}{x}$ for all $x \neq 0$.

Thus we can write:

$$\int \frac{1}{x}\, dx = \ln|x| + C.$$

Notice that this fills in the gap in the integration rule:

$$\int x^n \, dx = \frac{x^{n+1}}{n+1} + C\,; \quad n \neq -1$$

Since we know:

$$\int \frac{1}{x}\, dx = \ln|x| + C.$$

Thus, anytime we have an integrand that can be written as a fraction where the numerator is "essentially" (i.e. up to a constant multiple) the derivative of the denominator and we can find the anti-derivative by letting u equal the denominator.

Ex. Evaluate the following: $\int \frac{x}{3+x^2}\, dx.$

Notice that the numerator is the derivative of the denominator – except for a factor of 2.

Let $u = 3 + x^2$

$du = 2x\, dx$

$\frac{1}{2} du = x\, dx$

$$\int \frac{x}{3+x^2}\, dx = \int \left(\frac{1}{2}\right)\frac{1}{u} du = \frac{1}{2}\int \frac{1}{u} du = \frac{1}{2}\ln|u| + C$$

$$= \frac{1}{2}\ln|3 + x^2| + C.$$

Ex. Evaluate the following: $\int \frac{x-1}{x^2-2x+4}\, dx.$

Notice that $\frac{d}{dx}(x^2-2x+4) = 2x-2 =$ twice the numerator.

Let $u = x^2 - 2x + 4$

$du = (2x-2)\, dx$

$\frac{1}{2} du = (x-1)\, dx$

Substituting into the integral we get:

$$\int \frac{x-1}{x^2-2x+4}\, dx = \int \frac{\frac{1}{2}}{u} du = \frac{1}{2}\int \frac{1}{u}\, du$$

$$= \frac{1}{2}\ln|u| + C = \frac{1}{2}\ln|x^2-2x+4| + C.$$

Ex. Evaluate the following: $\int \frac{\cos x+1}{\sin x+x}\, dx.$

Notice that $\frac{d}{dx}(\sin x + x) = \cos x + 1$

Let $u = \sin x + x$

$du = (\cos x + 1)\, dx$

$$\int \frac{\cos x + 1}{\sin x + x}\, dx = \int \frac{du}{u}$$

$$= \ln|u| + C = \ln|\sin x + x| + C.$$

Ex. Evaluate the following: $\int \cot x \ dx$.

$\int \cot x \ dx = \int \frac{\cos x}{\sin x} \ dx$

$\frac{d}{dx}(\sin x) = \cos x$

Let $u = \sin x$

$du = \cos x \, dx$

$$\int \cot x \ dx = \int \frac{\cos x}{\sin x} \ dx = \int \frac{du}{u}$$

$$= \ln|u| + C = \ln|\sin x| + C.$$

Ex. Evaluate the following: $\int_1^{e^2} \frac{\ln x}{x} \ dx = \int_1^{e^2} (\ln x)\left(\frac{1}{x}\right) \ dx$.

Let $u = \ln x$; when $x = 1$, $u = \ln 1 = 0$

$du = \frac{1}{x} dx$; when $x = e^2$, $u = \ln e^2 = 2 \ln e = 2$

$$\int_{x=1}^{x=e^2} (\ln x)\left(\frac{1}{x}\right) dx = \int_{u=0}^{u=2} u \, du$$

$$= \frac{u^2}{2}\bigg|_{u=0}^{u=2} = \frac{4}{2} - \frac{0}{2} = 2.$$

Logarithmic Differentiation

Sometimes calculating derivatives of messy functions that involve products, quotients, or powers can be simplified by first taking logarithms of both sides of the equation, then differentiating. This is called **logarithmic differentiation**. However, remember to differentiate **both** sides of the equation.

Ex. Find $\frac{dy}{dx}$ for the function $y = \frac{\left(x^4+2\right)^5 \cos x}{\sqrt{x^2+1}}$.

Step 1: Take the natural log of both sides

$$\ln y = \ln\left(\frac{\left(x^4+2\right)^5 \cos x}{\sqrt{x^2+1}}\right)$$

Step 2: Use logarithm laws to expand one side. From earlier we saw:

$$\ln y = 5\ln(x^4+2) + \ln\cos x - \frac{1}{2}\ln(x^2+1)$$

Step 3: Differentiate both sides of the equation with respect to x.

$$\frac{1}{y}\frac{dy}{dx} = 5\left(\frac{1}{x^4+2}\right)(4x^3) + \frac{1}{\cos x}(-\sin x) - \frac{1}{2}\left(\frac{1}{x^2+1}\right)(2x)$$

$$\frac{1}{y}\frac{dy}{dx} = \frac{20x^3}{x^4+2} - \tan x - \frac{x}{x^2+1}$$

Step 4: Solve for $\frac{dy}{dx}$ as a function of x alone.

$$\frac{dy}{dx} = y\left[\frac{20x^3}{x^4+2} - \tan x - \frac{x}{x^2+1}\right]$$

$$= \frac{(x^4+2)^5 \cos x}{\sqrt{x^2+1}}\left[\frac{20x^3}{x^4+2} - \tan x - \frac{x}{x^2+1}\right].$$