Taylor Series and Maclaurin Series

Taylor series and Maclaurin series are power series representations of functions (Maclaurin series is a special case of Taylor series where the power series representation is around $a = 0$).

Suppose $f(x)$ has a power series representation around $x = a$:

$$f(x) = c_0 + c_1(x-a) + c_2(x-a)^2 + c_3(x-a)^3 + \cdots + c_n(x-a)^n + \cdots$$

for $|x - a| < R; \quad R > 0.$

Notice that at $x = a$ we get:

$$f(a) = c_0 + c_1(a-a) + c_2(a-a)^2 + c_3(a-a)^3 + \cdots + c_n(a-a)^n + \cdots$$

$\boxed{\boldsymbol{f(a) = c_0}}$.

Now let's calculate the derivatives of $f(x)$ at $x = a$:

$$f'(x) = c_1 + 2c_2(x-a) + 3c_3(x-a)^2 + \cdots + nc_n(x-a)^{n-1} + \cdots$$

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$\boxed{\boldsymbol{f'(a) = c_1}}$.

$$f''(x) = 2c_2 + 3\cdot 2c_3(x-a) + 4\cdot 3c_4(x-a)^2 + \ldots + n(n-1)c_n(x-a)^{n-2} + \cdots$$

$f''(a) = 2c_2$, which means that $\boxed{\boldsymbol{\frac{f''(a))}{2} = c_2}}$.

$$f'''(x) = 3 \cdot 2c_3 + 4 \cdot 3 \cdot 2c_4(x-a) + 5 \cdot 4 \cdot 3c_4(x-a)^2 + \cdots$$
$$+n(n-1)(n-2)c_n(x-a)^{n-3} + \cdots$$

$f'''(a) = 3 \cdot 2c_3$, which means that $\boxed{\boldsymbol{\frac{f'''(a)}{3!} = c_3}}$.

By the same reasoning:

$f^n(a) = n!\, c_n$, which means that $\boxed{\boldsymbol{\frac{f^n(a)}{n!} = c_n}}$.

Theorem: If f has a power series expansion around $x = a$,

$f(x) = \sum_{n=0}^{\infty} c_n(x-a)^n$; for $|x-a| < R$;

then $c_n = \frac{f^n(a)}{n!}$ so we know

$$\boldsymbol{f(x) = \sum_{n=0}^{\infty} \frac{f^n(a)}{n!}(x-a)^n}$$
$$\boldsymbol{= f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \cdots}$$
$$\boldsymbol{+\frac{f^n(a)}{n!}(x-a)^n + \cdots}$$

This is called the **Taylor series** of the function f around $x = a$.

For the special case when $a = 0$, the Taylor series becomes:

$$\boldsymbol{f(x) = \sum_{n=0}^{\infty} \frac{f^n(0)}{n!} x^n}$$
$$\boldsymbol{= f(0) + f'(0)(x) + \frac{f''(0)}{2!}(x)^2 + \frac{f'''(0)}{3!}(x)^3 + \cdots + \frac{f^n(0)}{n!}(x)^n + \cdots}$$

This is called the **Maclaurin series** of the function f.

Ex. Find the Maclaurin series for $f(x) = e^x$ (You need to know this series).

To find a Maclaurin series, we need to find f and all of its derivatives at $x = 0$ (for a general Taylor series around $x = a$ we would need to find f and its derivatives at $x = a$ and plug into the Taylor series formula).

$f(x) = e^x$ $\qquad f(0) = e^0 = 1$

$f'(x) = e^x$ $\qquad f'(0) = e^0 = 1$

$f''(x) = e^x$ $\qquad f''(0) = e^0 = 1$

$f'''(x) = e^x$ $\qquad f'''(0) = e^0 = 1$

$\vdots$

$f^n(x) = e^x$ $\qquad f^n(0) = e^0 = 1$

Now we plug into the Maclaurin series formula:

$$f(x) = f(0) + f'(0)(x) + \frac{f''(0)}{2!}(x)^2 + \cdots + \frac{f^n(0)}{n!}(x)^n + \cdots$$

$$= 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots + \frac{x^n}{n!} + \cdots = \sum_{n=0}^{\infty} \frac{x^n}{n!} = e^x.$$

Let's find the radius of convergence of the Maclaurin series for e^x:

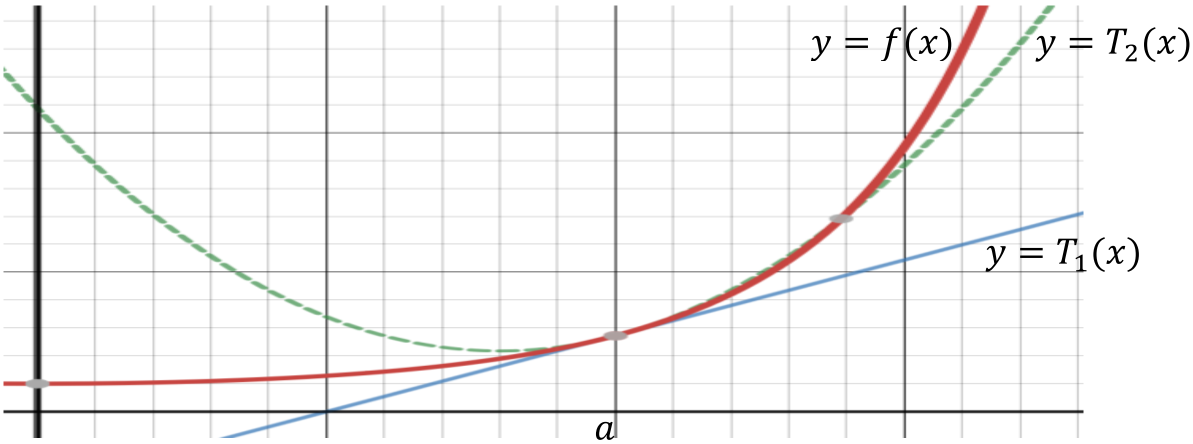
$$R = \lim_{n\to\infty} \left| \frac{x^{n+1}}{(n+1)!} \cdot \frac{n!}{x^n} \right| = \lim_{n\to\infty} \left| \frac{x}{n+1} \right| = 0 \text{ ; for all values of } x.$$

Thus, $R = \infty$ and $\sum_{n=0}^{\infty} \frac{x^n}{n!}$ converges for all values of x.

So if $f(x) = e^x$ has a power series expansion about $x = 0$,

then $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$.

A Taylor Series (or Maclaurin Series) is a generalization of the linear approximation:



$$T_1(x) = f(a) + f'(a)(x - a)$$

$$T_2(x) = f(a) + f'(a)(x - a) + \frac{f''(a)}{2!}(x - a)^2$$

$$T_3(x) = f(a) + f'(a)(x - a) + \frac{f''(a)}{2!}(x - a)^2 + \frac{f'''(a)}{3!}(x - a)^3$$

$$\vdots$$

$$T_n(x) = f(a) + f'(a)(x - a) + \frac{f''(a)}{2!}(x - a)^2 + \frac{f'''(a)}{3!}(x - a)^3$$

$$+ \cdots + \frac{f^n(a)}{n!}(x - a)^n .$$

In general, $f(x)$ is equal to its Taylor Series if:

$$f(x) = \lim_{n \to \infty} T_n(x) .$$

The polynomials, $T_1, T_2, T_3, \ldots, T_n$ are called **Taylor Polynomials**.

Ex. Find the Taylor Polynomials T_1, T_2, T_3, and T_n for $f(x) = e^x$ around $x = 0$.

Since $f^i(0) = 1$, for $i = 0,1,2, \dots$ we have:

$$T_1(x) = f(0) + f'(0)(x) = 1 + x$$

$$T_2(x) = f(0) + f'(0)(x) + \frac{f''(0)}{2!}(x)^2 = 1 + x + \frac{x^2}{2}$$

$$T_3(x) = f(0) + f'(0)(x) + \frac{f''(0)}{2!}(x)^2 + \frac{f'''(0)}{3!}(x)^3$$

$$= 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!}$$

$$T_n(x) = f(0) + f'(0)(x) + \frac{f''(0)}{2!}(x)^2 + \cdots + \frac{f^n(0)}{n!}(x)^n$$

$$= 1 + x + \frac{x^2}{2!} + \cdots + \frac{x^n}{n!}.$$

Let $R_n(x) = f(x) - T_n(x)$. $R_n(x)$ is called the **Remainder** of the Taylor series. If we can show that $\lim_{n\to\infty} R_n(x) = 0$, then we have:

$$f(x) = \lim_{n\to\infty} T_n(x)$$

and the Taylor Series converges to the function.

Theorem: If $f(x)$ has $n+1$ derivatives in an interval I that contains $x = a$, then for x in I there is a number z between x and "a" such that:

$$R_n(x) = \frac{f^{n+1}(z)}{(n+1)!}(x-a)^{n+1} \ .$$

1. Notice that the RHS is close to the $(n+1)^{st}$ order term of the Taylor series $\frac{f^{n+1}(a)}{(n+1)!}(x-a)^{n+1}$.
2. $R_n(x) = \frac{f^{n+1}(z)}{(n+1)!}(x-a)^{n+1}$ is called the **Lagrange form of the remainder**.

Ex. Show that for the function $f(x) = e^x$, $\lim_{n\to\infty} R_n(x) = 0$ for all real values of x, where $R_n(x)$ is the remainder of the Taylor polynomials around $x = 0$.

Since we are using Taylor Polynomials around $x = 0$, the Lagrange form of the remainder is: $R_n(x) = \frac{f^{n+1}(z)}{(n+1)!}(x)^{n+1}$.

We need to show that $\lim_{n\to\infty} R_n(x) = 0$ for all real values of x.

Case 1: $x > 0$; Since $f(x) = e^x$, $f^{(n)}(x) = e^x$.

Thus, $f^{(n)}(z) = e^z$, where $0 < z < x$.

So we have:

$$0 < R_n(x) = \frac{e^z}{(n+1)!}(x)^{n+1} < \frac{e^x}{(n+1)!}(x)^{n+1}.$$

Notice for any fixed number x,

$\lim_{n\to\infty} \frac{x^n}{n!} = 0$; so we can say:

$\lim_{n\to\infty} \frac{e^x x^n}{n!} = 0$; thus by the squeeze theorem

$\lim_{n\to\infty} R_n(x) = 0.$

Case 2: $x < 0$; so now $x < z < 0,$ which means that $e^z < e^0 = 1.$

So we have:

$$0 < \left|\frac{e^z}{(n+1)!}(x)^{n+1}\right| < \left|\frac{x^{n+1}}{(n+1)!}\right| .$$

Once again we know $\lim_{n\to\infty} \frac{x^n}{n!} = 0$, so by the squeeze theorem

$\lim_{n\to\infty} R_n(x) = 0.$

Thus, $f(x) = e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$ for all real values of x.

In particular, for $x = 1$ we get the following amazing series:

$$e^1 = e = \sum_{n=0}^{\infty} \frac{(1)^n}{n!} = 1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \frac{1}{5!} + \cdots \frac{1}{n!} + \cdots$$

Ex. Find the Maclaurin series for $f(x) = \cos(x)$ and show that it equals $\cos x$ for all x.

To find a Maclaurin (or Taylor) series we have to find an expression for the n^{th} derivative at $x = 0$ (or $x = a$ for a general Taylor series).

In this case, there is a pattern in the derivatives of $\cos x$, as well as $\sin x$.

$f(x) = \cos(x)$ $\quad\quad f(0) = 1$

$f'(x) = -\sin(x)$ $\quad\quad f'(0) = 0$

$f''(x) = -\cos(x)$ $\quad\quad f''(0) = -1$

$f'''(x) = \sin(x)$ $\quad\quad f'''(0) = 0$

$f^4(x) = \cos(x)$ $\quad\quad f^4(0) = 1$

So the odd derivatives at $x = 0$ are equal to 0 and the even derivatives, i.e. the $(2n)^{th}$ derivative is equal to $(-1)^n$.

Now let's plug into the Maclaurin series formula:

$$f(x) = f(0) + f'(0)(x) + \frac{f''(0)}{2!}(x)^2 + \frac{f'''(0)}{3!}(x)^3$$

$$+ \cdots + \frac{f^n(0)}{n!}(x)^n + \cdots$$

$$= 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots + (-1)^n \frac{x^{2n}}{(2n)!} + \cdots$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}.$$

Now let's show that this series converges to $\cos x$ for all real numbers.

To do this, we must show that $\lim_{n\to\infty} R_n(x) = 0$ for all real numbers x.

Since we are using a Maclaurin series, i.e. "a"$= 0$, the remainder has the form:

$$R_n(x) = \frac{f^{n+1}(z)}{(n+1)!}(x)^{n+1} \;\; ; \text{ where } z \text{ is between 0 and } x.$$

Notice that every derivative of $f(x) = \cos(x)$ is either $\pm\cos(x)$ or $\pm\sin(x)$.

In every case, we have $|f^k(z)| \le 1$.

Thus we have:

$$0 \le |R_n(x)| = \left|\frac{f^{n+1}(z)}{(n+1)!}(x)^{n+1}\right| \le \left|\frac{x^{n+1}}{(n+1)!}\right|.$$

Since $\lim_{n\to\infty} \frac{x^n}{n!} = 0$, by the squeeze theorem we have $\lim_{n\to\infty} R_n(x) = 0$ for all real values of x.

Thus, we have shown that the Maclaurin series converges to the function and:

$$\cos(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots + (-1)^n \frac{x^{2n}}{(2n)!} + \cdots$$
$$= \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!} \, .$$

You must know this series as well as the one for $\sin x$!!

Ex. Find the Maclaurin series for $f(x) = \sin(x)$.

We could find this series the same way we did for $\cos(x)$, but it's easier to just differentiate the series for $\cos(x)$ and multiply by -1.

$$f(x) = \cos(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots + (-1)^n \frac{x^{2n}}{(2n)!} + \cdots$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}.$$

$$f'(x) = -\sin(x) = -x + \frac{x^3}{3!} - \frac{x^5}{5!} + \frac{x^7}{7!} + \cdots + (-1)^n \frac{x^{2n-1}}{(2n-1)!} + \cdots$$

$$\sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots + (-1)^{(n-1)} \frac{x^{2n-1}}{(2n-1)!}$$

$$+(-1)^{(n)} \frac{x^{2n+1}}{(2n+1)!} + \cdots$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}.$$

We can use the Maclaurin (or Taylor) series of known functions like e^x, $\sin x$, or $\cos x$ to find series for related functions.

Ex. Find the Maclaurin series for $f(x) = \frac{e^x - 1}{x}$ and $g(x) = e^{-x^2}$.

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots + \frac{x^n}{n!} + \cdots$$

$$e^x - 1 = x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots + \frac{x^n}{n!} + \cdots$$

$$\frac{e^x - 1}{x} = 1 + \frac{x}{2!} + \frac{x^2}{3!} + \frac{x^3}{4!} + \cdots + \frac{x^{n-1}}{n!} + \cdots = \sum_{n=1}^{\infty} \frac{x^{n-1}}{n!}.$$

Now to find $g(x) = e^{-x^2}$, just substitute $-x^2$ into the series for e^x.

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots + \frac{x^n}{n!} + \cdots$$

$$e^{-x^2} = 1 + (-x^2) + \frac{(-x^2)^2}{2!} + \frac{(-x^2)^3}{3!} + \frac{(-x^2)^4}{4!} + \cdots + \frac{(-x^2)^n}{n!} + \cdots$$

$$= 1 - x^2 + \frac{x^4}{2!} - \frac{x^6}{3!} + \frac{x^8}{4!} + \cdots + \frac{(-1)^n x^{2n}}{n!} + \cdots = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{n!}.$$

Ex. Find the Taylor series for $f(x) = e^{ix}$ and show $e^{ix} = \cos x + i \sin x$, known as Euler's Formula.

$$e^{ix} = 1 + (ix) + \frac{(ix)^2}{2!} + \frac{(ix)^3}{3!} + \frac{(ix)^4}{4!} + \cdots + \frac{(ix)^n}{n!} + \cdots$$

$$= 1 + ix - \frac{x^2}{2!} - \frac{i^3 x^3}{3!} + \frac{x^4}{4!} + \cdots + \frac{i^n x^n}{n!} + \cdots$$

$$= 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots + i\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots\right)$$

$$= \cos x + i \sin x .$$

Notice that at $x = \pi$ we get:

$$e^{\pi i} = cos\pi + isin\pi = -1 \quad \Longrightarrow \quad \boxed{e^{\pi i} + 1 = 0}$$

Ex. Find the Maclaurin series for $f(x) = \frac{\sin(x)-x}{x^3}$.

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots + \frac{(-1)^n x^{2n+1}}{(2n+1)!} + \cdots$$

$$\sin x - x = -\frac{x^3}{3!} + \frac{x^5}{5!} + \cdots + \frac{(-1)^n x^{2n+1}}{(2n+1)!} + \cdots$$

$$\frac{\sin x - x}{x^3} = \frac{-\frac{x^3}{3!} + \frac{x^5}{5!} + \cdots + \frac{(-1)^n x^{2n+1}}{(2n+1)!} + \cdots}{x^3}$$

$$= -\frac{1}{3!} + \frac{x^2}{5!} - \frac{x^4}{7!} \ldots + \frac{(-1)^n x^{2n-2}}{(2n+1)!} + \cdots$$

$$= \sum_{n=1}^{\infty} \frac{(-1)^n x^{2n-2}}{(2n+1)!}.$$

Ex. Find the Taylor series for $f(x) = \sin(x)$ around $a = \pi$.

$$f(x) = \sin(x) \qquad f(\pi) = 0$$
$$f'(x) = \cos(x) \qquad f'(\pi) = -1$$
$$f''(x) = -\sin(x) \qquad f''(\pi) = 0$$
$$f'''(x) = -\cos(x) \qquad f'''(\pi) = 1$$
$$f^4(x) = \sin(x) \qquad f^4(\pi) = 0$$

$$f(x) = f(\pi) + f'(\pi)(x-\pi) + \frac{f''(\pi)}{2!}(x-\pi)^2$$
$$+ \cdots + \frac{f^n(\pi)}{n!}(x-\pi)^n + \cdots$$

Since $f^{2k}(\pi) = 0$ we have:

$$= -(x-\pi) + \frac{1}{3!}(x-\pi)^3 - \frac{1}{5!}(x-\pi)^5$$
$$+ \cdots + \frac{(-1)^{n+1}}{(2n+1)!}(x-\pi)^{2n+1} + \cdots$$

$$f(x) = \sum_{n=0}^{\infty} \frac{(-1)^{n+1}}{(2n+1)!}(x-\pi)^{2n+1}.$$

Ex. Find the Maclaurin series for $f(x) = (1+x)^k$, where k is a real number.

$f(x) = (1+x)^k$

$f'(x) = k(1+x)^{k-1}$

$f''(x) = k(k-1)(1+x)^{k-2}$

$\vdots$

$f^{(n)}(x) = k(k-1)\cdots(k-n+1)(1+x)^{k-n}$

$f(0) = 1$

$f'(0) = k$

$f''(0) = k(k-1)$

$\vdots$

$f^{n}(0) = k(k-1)\cdots(k-n+1)$

$$f(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \cdots \frac{f^{n}(0)}{n!}x^n + \cdots$$

$$(1+x)^k = 1 + kx + \frac{k(k-1)}{2!}x^2 + \cdots + \frac{k(k-1)\ldots(k-n+1)}{n!}x^n + \cdots$$

Ex. Find the Maclaurin series for $\frac{x^2}{\sqrt{4+x}}$.

$$\frac{1}{\sqrt{4+x}} = (4+x)^{-\frac{1}{2}} = 4^{-\frac{1}{2}}\left(1+\frac{x}{4}\right)^{-\frac{1}{2}} = \frac{1}{2}\left(1+\frac{x}{4}\right)^{-\frac{1}{2}}$$

notice this is similar to $(1+x)^k$, $k = -\frac{1}{2}$.

$$\left(1+\frac{x}{4}\right)^{-\frac{1}{2}} = 1 - \frac{1}{2}\left(\frac{x}{4}\right) + \frac{1}{2}\left(\frac{3}{2}\right)\left(\frac{x}{4}\right)^2 + \cdots$$
$$+ \left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\ldots\left(-\frac{1}{2} - n + 1\right)\left(\frac{x}{4}\right)^n + \cdots$$

$$\frac{x^2}{\sqrt{4+x}} = \frac{1}{2}[x^2 - \frac{1}{2}\left(\frac{x^3}{4}\right) + \frac{1}{2}\left(\frac{3}{2}\right)\left(\frac{x^4}{4^2}\right) + \cdots$$
$$+ \left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\ldots\left(-\frac{1}{2} - n + 1\right)\left(\frac{x^{n+2}}{4^n}\right) + \cdots].$$

Ex. Evaluate $\int_0^1 e^{-x^2}dx$ using a Maclaurin series. Approximate $\int_0^1 e^{-x^2}dx$ to within 0.001 .

$$\int_0^1 e^{-x^2}dx = \int_0^1 \left(1 - x^2 + \frac{x^4}{2!} - \frac{x^6}{3!} + \frac{x^8}{4!} + \cdots\right)dx$$

$$= x - \frac{x^3}{3} + \frac{x^5}{5(2!)} - \frac{x^7}{7(3!)} + \frac{x^9}{9(4!)} + \cdots \Big|_0^1$$

$$= 1 - \frac{1}{3} + \frac{1}{5(2!)} - \frac{1}{7(3!)} + \frac{1}{9(4!)} - \frac{1}{11(5!)} + \cdots$$

This is an alternating series so the error after n terms is less than the absolute value of the $(n+1)^{st}$ term.

Notice that $\frac{1}{11(5!)} = \frac{1}{1320} < 0.001$ so:

$$\int_0^1 e^{-x^2}dx \approx 1 - \frac{1}{3} + \frac{1}{5(2!)} - \frac{1}{7(3!)} + \frac{1}{9(4!)} \approx 0.7475$$

with an error of less than 0.001 .

Ex. Use Maclaurin series to find $\lim_{x\to 0} \frac{\cos(x^3)-1+(.5)x^6}{x^{12}}$.

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots$$

$$\cos(\,x^3) = 1 - \frac{(x^3)^2}{2!} + \frac{(x^3)^4}{4!} - \frac{(x^3)^6}{6!} + \cdots$$

$$= 1 - \frac{x^6}{2!} + \frac{x^{12}}{4!} - \frac{x^{18}}{6!} + \cdots$$

$$\lim_{x\to 0} \frac{\cos(x^3) - 1 + \frac{1}{2}x^6}{x^{12}} = \lim_{x\to 0} \frac{\frac{x^{12}}{4!} - \frac{x^{18}}{6!} + \cdots}{x^{12}}$$

$$= \lim_{x\to 0} \left(\frac{1}{4!} - \frac{x^6}{6!} + \cdots \right) = \frac{1}{4!} = \frac{1}{24} \,.$$

Power series can be added, subtracted, multiplied and divided much like polynomials.

Ex. Find the first 3 non-zero terms in the Maclaurin series for:

a. $(e^x)[\ln(1-x)]$

b. $\dfrac{x}{\sin(x)}$

a. $$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$

$$\ln(1-x) = x + \frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{4} + \cdots$$

$$(e^x)(\ln(1-x)) = \left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots\right)\left(x + \frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{4} + \cdots\right)$$

$$= x + \left(x^2 + \frac{x^2}{2}\right) + \left(\frac{x^3}{2} + \frac{x^3}{2} + \frac{x^3}{3}\right) + \cdots$$

$$= x + \frac{3}{2}x^2 + \frac{4}{3}x^3 + \cdots$$

b. $\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$

$$\frac{x}{\sin x} = \frac{x}{x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots} = \frac{x}{x\left(1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \frac{x^6}{7!} + \cdots\right)}$$

$$= \frac{1}{1 - \left(\frac{x^2}{3!} - \frac{x^4}{5!} + \frac{x^6}{7!} + \cdots\right)}$$

$$= 1 + \left(\frac{x^2}{3!} - \frac{x^4}{5!} + \frac{x^6}{7!} + \cdots\right) + \left(\frac{x^2}{3!} - \frac{x^4}{5!} + \frac{x^6}{7!} + \cdots\right)^2 + \cdots$$

$$= 1 + \frac{x^2}{3!} - \frac{x^4}{5!} + \left(\frac{x^2}{3!} - \frac{x^4}{5!} + \cdots\right)\left(\frac{x^2}{3!} - \frac{x^4}{5!} + \cdots\right) + \cdots$$

$$= 1 + \frac{x^2}{3!} - \frac{x^4}{5!} + \frac{x^4}{36} + \cdots$$

$$= 1 + \frac{x^2}{6} + \left(\frac{1}{36} - \frac{1}{120}\right)x^4 + \cdots$$

$$= 1 + \frac{x^2}{6} + \frac{7}{360}x^4 + \cdots$$