Surface Integrals of Vector Fields

The notion of Work motivated the definition of the line integral of a vector field. The notion of Flux motivates the definition of the surface integral of a vector field. Flux measures the rate at which a gas or fluid crosses a surface. This is given by the integral of a velocity vector field $\vec{F}$ over a surface S.

Def. Let $\vec{F}$ be a vector field defined on a surface S, parametrized by $\vec{\Phi}$ then

$$\iint_{\boldsymbol{S}} \vec{\boldsymbol{F}} \cdot \boldsymbol{d}\vec{\boldsymbol{S}} = \iint_{\boldsymbol{D}} \vec{\boldsymbol{F}}(\vec{\boldsymbol{\Phi}}(\boldsymbol{u}, \boldsymbol{v})) \cdot (\vec{\boldsymbol{T}}_{\boldsymbol{u}} \times \vec{\boldsymbol{T}}_{\boldsymbol{v}})\boldsymbol{dudv}.$$

Ex. Find the flux of the vector field $\vec{F}(x, y, z) = z\vec{\imath} + y\vec{\jmath} + x\vec{k}$ across the unit sphere $x^2 + y^2 + z^2 = 1$.

Let's start with a standard parametrization of the sphere (outward pointing normal):

$$\vec{\Phi}(\phi, \theta) = (cos\theta sin\phi)\vec{\imath} + (sin\theta sin\phi)\vec{\jmath} + (cos\phi)\vec{k};$$

$$0 \le \phi \le \pi, \;\; 0 \le \theta \le 2\pi.$$

$$\vec{T}_\phi = (cos\theta cos\phi)\vec{\imath} + (sin\theta cos\phi)\vec{\jmath} - (sin\phi)\vec{k}$$

$$\vec{T}_\theta = -(sin\theta sin\phi)\vec{\imath} + (cos\theta sin\phi)\vec{\jmath}$$

$$\vec{T}_\phi \times \vec{T}_\theta = \begin{vmatrix} \vec{\imath} & \vec{\jmath} & \vec{k} \\ cos\theta cos\phi & sin\theta cos\phi & -sin\phi \\ -sin\theta sin\phi & cos\theta sin\phi & 0 \end{vmatrix}$$

$$= cos\theta(sin^2\phi)\vec{\imath} + sin\theta(sin^2\phi)\vec{\jmath} + (cos^2\theta + sin^2\theta)(sin\phi cos\phi)\,\vec{k}$$

$$= cos\theta(sin^2\phi)\vec{\imath} + sin\theta(sin^2\phi)\vec{\jmath} + (sin\phi cos\phi)\,\vec{k}$$

$$\vec{F}\left(\vec{\Phi}(\phi,\theta)\right) = (cos\phi)\vec{\imath} + (sin\theta sin\phi)\vec{\jmath} + (cos\theta sin\phi)\vec{k}$$

$$\begin{aligned}\vec{F}\cdot\left(\vec{T}_\phi \times \vec{T}_\theta\right) &= cos\,\theta\,(\sin^2\phi)cos\,\phi + \sin^2\theta\,(\sin^3\phi) + cos\theta(\sin^2\phi)cos\phi \\ &= 2cos\,\theta\,(\sin^2\phi)cos\,\phi + \sin^2\theta\,(\sin^3\phi)\end{aligned}$$

$$\iint_S\ \vec{F}\cdot d\vec{S} = \iint_D\ \vec{F}(\vec{\Phi}(\phi,\theta))\cdot(\vec{T}_\phi \times \vec{T}_\theta)d\phi d\theta$$

$$= \int_{\theta=0}^{\theta=2\pi}\int_{\phi=0}^{\phi=\pi}[2cos\,\theta\,(\sin^2\phi)cos\,\phi + \sin^2\theta\,(\sin^3\phi)]\,d\phi d\theta$$

$$= 2\int_{\theta=0}^{\theta=2\pi} cos\theta d\theta \int_{\phi=0}^{\phi=\pi}\sin^2\phi(cos\phi)d\phi$$

$$+\int_{\theta=0}^{\theta=2\pi}\sin^2\theta d\theta\int_{\phi=0}^{\phi=\pi}\sin^3\phi d\phi.$$

$\int_{\theta=0}^{\theta=2\pi} cos\theta d\theta = 0$, so the first term is equal to 0.

$$\int_{\theta=0}^{\theta=2\pi}\sin^2\theta d\theta = \int_{\theta=0}^{\theta=2\pi}\left(\frac{1}{2}-\frac{1}{2}cos2\theta\right)d\theta = (\frac{1}{2}\theta - \frac{1}{4}sin2\theta)\Big|_0^{2\pi} = \pi$$

$\int_{\phi=0}^{\phi=\pi}\sin^3\phi d\,\phi = \int_{\phi=0}^{\phi=\pi} sin\phi\,(1-\cos^2\phi)d\,\phi$; now let $u = cos\phi$ to get

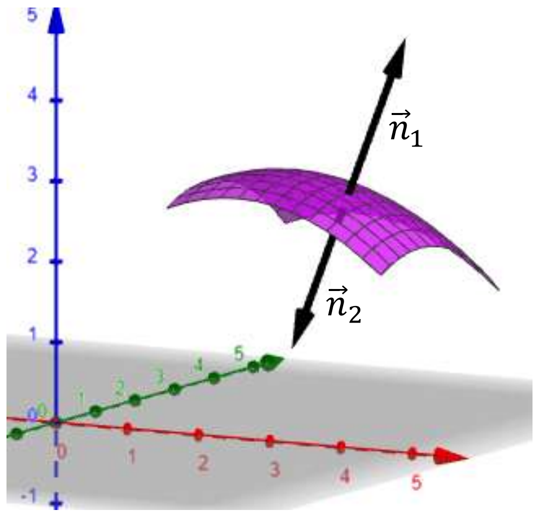
$$= -\int_{u=1}^{u=-1}(1-u^2)du = \frac{4}{3}.$$

Putting these three integrals together we have:

$$Flux = \iint_S\ \vec{F}\cdot d\vec{S} = \frac{4\pi}{3}\,.$$

Orientation

Def. An **oriented surface** S is a 2-sided surface with one side specified as the outside or positive side and the other side the inside or negative side. At each point $(x, y, z) \in S$ there are 2 unit normals, $\vec{n}_1$ and $\vec{n}_2$, where $\vec{n}_1 = -\vec{n}_2$.



Ex. Let's take the unit sphere $x^2 + y^2 + z^2 = 1$.

$$\vec{\Phi}(\phi, \theta) = (cos\theta sin\phi)\vec{\imath} + (sin\theta sin\phi)\vec{\jmath} + (cos\phi)\vec{k}; \quad 0 \leq \phi \leq \pi, \ \ 0 \leq \theta \leq 2\pi$$

As we just saw:

$$\vec{T}_\phi \times \vec{T}_\theta = cos\theta(sin^2\phi)\vec{\imath} + sin\theta(sin^2\phi)\vec{\jmath} + (sin\phi cos\phi)\,\vec{k}\,.$$

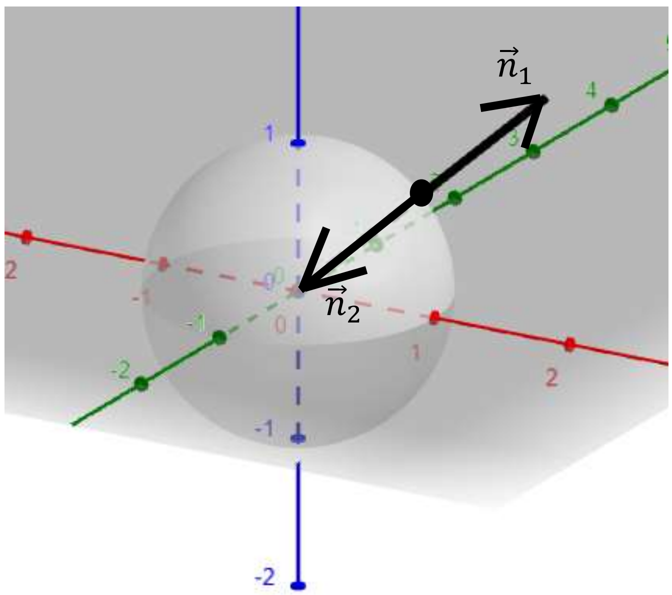
Thus we can calculate a unit normal vector by $\vec{n}_1 = \frac{\vec{T}_\phi \times \vec{T}_\theta}{|\vec{T}_\phi \times \vec{T}_\theta|}$.

$$|\vec{T}_\phi \times \vec{T}_\theta| = \sqrt{cos^2\theta(sin^4\phi) + sin^2\theta(sin^4\phi) + sin^2\phi cos^2\phi}$$

$$= sin\phi.$$

$$\vec{n}_1 = \frac{\vec{T}_\phi \times \vec{T}_\theta}{|\vec{T}_\phi \times \vec{T}_\theta|} = cos\theta(sin\phi)\vec{\imath} + sin\theta(sin\phi)\vec{\jmath} + (cos\phi)\,\vec{k} = x\vec{\imath} + y\vec{\jmath} + z\vec{k}$$

$$\vec{n}_2 = \frac{\vec{T}_\theta \times \vec{T}_\phi}{|\vec{T}_\theta \times \vec{T}_\phi|} = -cos\theta(sin\phi)\vec{\imath} - sin\theta(sin\phi)\vec{\jmath} - (cos\phi)\,\vec{k} = -x\vec{\imath} - y\vec{\jmath} - z\vec{k}.$$



All we have done here is switch which is the first variable and which is the second. In the first case we have (ϕ, θ), in the second case we have (θ, ϕ). Thus,

$$\vec{\Phi}(\phi, \theta) = (cos\theta sin\phi)\vec{\imath} + (sin\theta sin\phi)\vec{\jmath} + (cos\phi)\vec{k}; \quad 0 \le \phi \le \pi, \;\; 0 \le \theta \le 2\pi$$

and

$$\vec{\Phi}(\theta, \phi) = (cos\theta sin\phi)\vec{\imath} + (sin\theta sin\phi)\vec{\jmath} + (cos\phi)\vec{k}; \quad 0 \le \phi \le \pi, \;\; 0 \le \theta \le 2\pi$$

are both parametrizations of the unit sphere, but they have different orientations.

Notice that $\vec{n}_1$ points outward (i.e. positive orientation) and $\vec{n}_2$ points inward (i.e. negative orientation).

<u>Orientation when S is given by $z = g(x, y)$.</u>

When a surface, S, is given by $z = g(x, y)$, we can always parametrize it by:

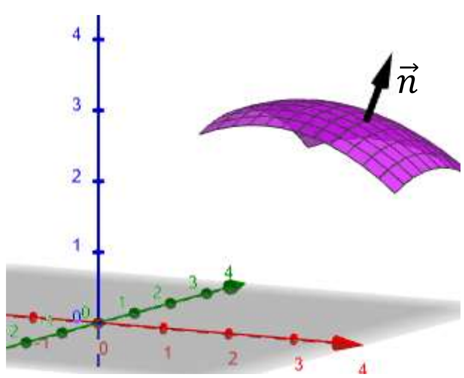
$x = u \quad y = v \quad z = g(u, v);$ ie $\vec{\Phi}(u, v) = < u, v, g(u, v) >$

$\vec{T}_u = < 1,0, g_u > \qquad \vec{T}_v = < 0,1, g_v >$

$\vec{T}_u \times \vec{T}_v = \begin{vmatrix} \vec{\imath} & \vec{\jmath} & \vec{k} \\ 1 & 0 & g_u \\ 0 & 1 & g_v \end{vmatrix} = -(g_u)\vec{\imath} - (g_v)\vec{\jmath} + \vec{k};$ Thus a unit normal is:

$$\vec{n} = \frac{\vec{T}_u \times \vec{T}_v}{|\vec{T}_u \times \vec{T}_v|} = \frac{-(g_u)\vec{\imath} - (g_v)\vec{\jmath} + \vec{k}}{\sqrt{1 + (g_u)^2 + (g_v)^2}}.$$

Notice that the k component is positive so the unit normal has an "upward" component. This will be taken as the positive orientation for this surface.



Theorem: Let S be an oriented surface and let $\vec{\Phi}_1$ and $\vec{\Phi}_2$ be two regular orientation preserving parametrizations, with $\vec{F}$ a continuous vector field on S, then:

$$\iint_{\vec{\Phi}_1} \vec{F} \cdot d\vec{S} = \iint_{\vec{\Phi}_2} \vec{F} \cdot d\vec{S}.$$

If $\vec{\Phi}_1$ and $\vec{\Phi}_2$ have opposite orientations then

$$\iint_{\vec{\Phi}_1} \vec{F} \cdot d\vec{S} = -\iint_{\vec{\Phi}_2} \vec{F} \cdot d\vec{S}.$$

If f is a real valued continuous function defined on S and if $\vec{\Phi}_1$ and $\vec{\Phi}_2$ are parametrizations of S, then :

$$\iint_{\vec{\Phi}_1} f dS = \iint_{\vec{\Phi}_2} f dS.$$

<u>Relationship of Integrals of Vector Fields to Integrals of Scalar Functions</u>

Recall that for line integrals of vector fields we had:

$$\int_c \vec{F} \cdot d\vec{s} = \int_c \vec{F} \cdot \overrightarrow{c'}(t)dt$$

$$= \int_{t=a}^{t=b} \vec{F} \cdot \frac{\overrightarrow{c'}(t)}{\left|\overrightarrow{c'}(t)\right|}\left(\left|\overrightarrow{c'}(t)\right|\right)dt = \int_c \vec{F} \cdot \vec{T} ds$$

where $\vec{T} = \frac{\overrightarrow{c'}(t)}{\left|\overrightarrow{c'}(t)\right|}$ is the unit tangent vector to $\vec{c}$(t).

So we have:

$\int_c \vec{F} \cdot d\vec{s} = \int_c f ds$; where $f(x, y, z) = \vec{F} \cdot \vec{T}$. So f is the projection of the vector field $\vec{F}$ onto the unit tangent vector, $\vec{T}$, of the curve c.

Similarly, we can do the following with surface integrals of vector fields:

$$\iint_S \vec{F}\cdot d\vec{S} = \iint_D \vec{F}(\vec{\Phi}(u,v))\cdot(\vec{T}_u\times\vec{T}_v)dudv$$
$$= \iint_D \vec{F}\cdot\frac{\vec{T}_u\times\vec{T}_v}{|\vec{T}_u\times\vec{T}_v|}\left(|\vec{T}_u\times\vec{T}_v|\right)dudv$$
$$= \iint_D \vec{F}\cdot\vec{n}\left(|\vec{T}_u\times\vec{T}_v|\right)dudv = \iint_S \left(\vec{F}\cdot\vec{n}\right)dS.$$

So $\iint_S \vec{F}\cdot d\vec{S} = \iint_S f dS$; where $f(x,y,z) = \vec{F}\cdot\vec{n}$.

So $f(x,y,z)$ is the projection of the vector field $\vec{F}$ onto the unit normal vector, $\vec{n}$, of the surface S. This observation can sometimes help us to calculate surface integrals faster, particularly when $\vec{F}\cdot\vec{n}$ is a constant function.

Ex. Evaluate $\iint_S \vec{F}\cdot d\vec{S}$ where $\vec{F} = -(2x)\vec{\imath} - (2y)\vec{\jmath} - (2z)\vec{k}$, and S is the unit sphere (if no orientation is specified, use the “positive” orientation, ie, $\vec{n}$ pointing outward).

$$\iint_S \vec{F}\cdot d\vec{S} = \iint_S f dS; \text{ where } f(x,y,z) = \vec{F}\cdot\vec{n}$$

For the unit sphere $\vec{n} =< x,y,z >$; thus we have:

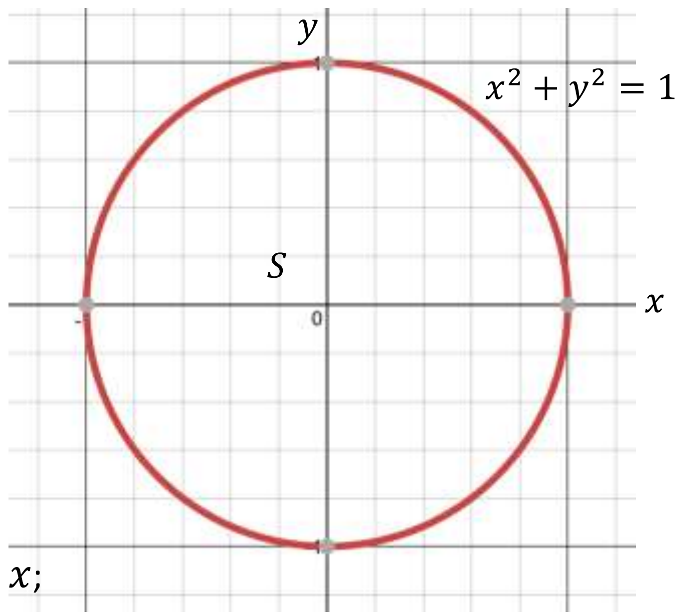
$$f(x,y,z) = \vec{F}\cdot\vec{n} =< -2x,-2y,-2z >\cdot< x,y,z >$$
$$= -2(x^2+y^2+z^2) = -2$$

since (x,y,z) lies on the unit sphere $x^2+y^2+z^2 = 1$.

$$\iint_S \vec{F}\cdot d\vec{S} = \iint_S f dS = \iint_S -2dS = -2\iint_S dS = -2(surface\ area\ of\ S)$$

$$= -2(4\pi r^2) = -8\pi.$$

Ex. Evaluate $\iint_S \vec{F}\cdot d\vec{S}$, where $\vec{F}(x,y,z) = x\vec{k}$, and S is the surface $x^2+y^2 \le 1$ in the x, y plane.



$\iint_S \vec{F}\cdot d\vec{S} = \iint_S f dS$;

where $f(x,y,z) = \vec{F}\cdot\vec{n}$.

In this case $\vec{n}$ is just the vector $\vec{k}$;

and $f(x,y,z) = \vec{F}\cdot\vec{n} = (x\vec{k})\cdot \vec{k} = x$;

$$\iint_S \vec{F}\cdot d\vec{S} = \iint_S f dS = \iint_S x dS = \iint_{x^2+y^2\le 1}(x)\, dydx\ .$$

Since we are integrating over the unit disk, change to polar coordinates:

$$= \int_{\theta=0}^{\theta=2\pi}\int_{r=0}^{r=1}(rcos\theta)(r)drd\theta$$

$$= \int_{\theta=0}^{\theta=2\pi} cos\theta d\theta \int_{r=0}^{r=1} r^2dr = (0)(\int_{r=0}^{r=1} r^2dr) = 0.$$

<u>Surface Integrals when S is given by $z = g(x, y)$</u>

If $z = g(x, y)$, we can always parametrize the surface by

$x = u, \quad y = v, \quad z = g(u, v)$, ie $\vec{\Phi}(u, v) = < u, v, g(u, v) >$.

$\vec{T}_u = < 1, 0, g_u > \qquad\qquad \vec{T}_v = < 0, 1, g_v >$

$$\vec{T}_u \times \vec{T}_v = \begin{vmatrix} \vec{\imath} & \vec{\jmath} & \vec{k} \\ 1 & 0 & g_u \\ 0 & 1 & g_v \end{vmatrix} = -g_u\vec{\imath} - g_v\vec{\jmath} + \vec{k}\ .$$

Since $x = u, \quad y = v$, we can use x and y instead of u and v. So we can write:

$\vec{T}_x \times \vec{T}_y = -g_x\vec{\imath} - g_y\vec{\jmath} + \vec{k} = < -g_x, -g_y, 1 >$

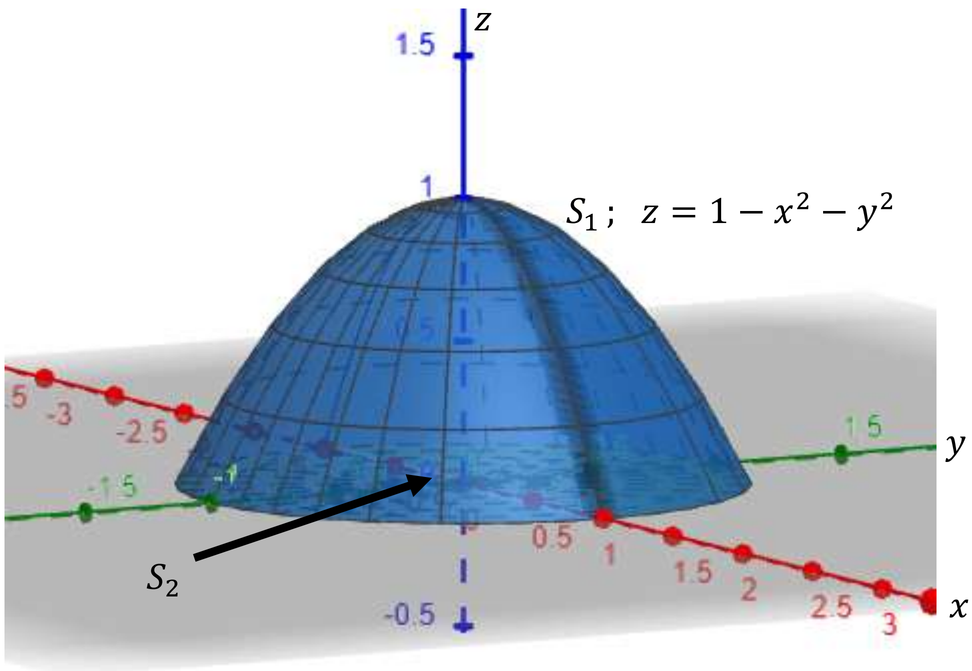
$\vec{F}(x, y, z) = (F_1)\vec{\imath} + (F_2)\vec{\jmath} + (F_3)k = < F_1, F_2, F_3 >$

$$\iint_S \vec{F} \cdot d\vec{S} = \iint_D \vec{F} \cdot (\vec{T}_x \times \vec{T}_y) dxdy$$

$$= \iint_D < F_1, F_2, F_3 > \cdot < -g_x, -g_y, 1 > dxdy$$

$$\boldsymbol{\iint_S \vec{F} \cdot d\vec{S} = \iint_D \left(-(F_1)g_x - (F_2)g_y + F_3\right) dxdy.}$$

Ex. Evaluate $\iint_S \vec{F} \cdot d\vec{S}$ where $\vec{F} = < y, x, z >$ and S is the boundary of the solid E given by the paraboloid $z = 1 - x^2 - y^2$ and the plane $z = 0$.



In this case, the surface S is made up of 2 pieces, S_1 and S_2. S_1 is the surface

$z = 1 - x^2 - y^2$ where $z \geq 0$, and S_2 is the surface in the xy-plane (ie $z = 0$) where $x^2 + y^2 \leq 1$.

$$\iint_S \vec{F} \cdot d\vec{S} = \iint_{S_1} \vec{F} \cdot d\vec{S} + \iint_{S_2} \vec{F} \cdot d\vec{S}$$

To calculate $\iint_{S_1} \vec{F} \cdot d\vec{S}$, notice that S_1 is given by $g(x, y) = 1 - x^2 - y^2$, $z \geq 0$.

Thus we can use the formula we just derived for $\iint_S \vec{F} \cdot d\vec{S}$ when S is given by

$z = g(x, y)$. In this case since $\vec{F} = < y, x, z >$, $F_1 = y$, $F_2 = x$, $F_3 = z$.

$g(x, y) = 1 - x^2 - y^2$; so $g_x = -2x$ and $g_y = -2y$.

$$\iint_{S_1} \vec{F} \cdot d\vec{S} = \iint_D \left(-(F_1)g_x - (F_2)g_y + F_3\right) dxdy$$

$$= \iint_D (2xy + 2xy + z)\, dxdy \, ; \quad \text{but } z = 1 - x^2 - y^2$$

$$= \iint_D (4xy + 1 - x^2 - y^2)\, dxdy.$$

D is the set where $g(x, y) = 1 - x^2 - y^2 \geq 0$, that is, $x^2 + y^2 \leq 1$.

Since D is the unit disk in the x, y plane, let's change to polar coordinates.

$$\iint_{S_1} \vec{F} \cdot d\vec{S} = \int_{\theta=0}^{\theta=2\pi} \int_{r=0}^{r=1} (4r^2 cos\theta sin\theta + 1 - r^2)(r) dr d\theta$$

$$= \int_{\theta=0}^{\theta=2\pi} \int_{r=0}^{r=1} (4r^3 cos\theta sin\theta + r - r^3) dr d\theta$$

$$= \int_{\theta=0}^{\theta=2\pi} r^4 cos\theta sin\theta + \frac{1}{2}r^2 - \frac{1}{4}r^4 \Big|_0^1 d\theta$$

$$= \int_{\theta=0}^{\theta=2\pi} \left(cos\theta sin\theta + \frac{1}{4}\right) d\theta = \frac{1}{4}(2\pi) = \frac{\pi}{2}$$

Since we can see $\int_{\theta=0}^{\theta=2\pi} (cos\theta sin\theta) d\theta = 0$ by letting $u = sin\theta$.

To calculate $\iint_{S_2} \vec{F} \cdot d\vec{S}$, notice that S_2 is a region in the x, y-plane so that $\vec{n} = -\vec{k}$ (the outward direction in this case is "down").

$$\iint_{S_2} \vec{F} \cdot d\vec{S} = \iint_{S_2} (\vec{F} \cdot \vec{n}) dS = \iint_{S_2} (< y, x, z > \cdot < 0,0,-1 >) dS$$

$$= \iint_{S_2} -z dS = 0, \quad \text{since } z = 0 \text{ on } S_2 \text{ (which is in the } xy\text{-plane)}$$

$$\iint_S \vec{F} \cdot d\vec{S} = \iint_{S_1} \vec{F} \cdot d\vec{S} + \iint_{S_2} \vec{F} \cdot d\vec{S} = \frac{\pi}{2} + 0 = \frac{\pi}{2}.$$

<u>Summary of formulas for Surface Integrals (of Scalar Functions and Vector Fields)</u>

1. For parametrized surfaces $\vec{\Phi}(u,v)$

a. Surface integral of a scalar function $f(x,y,z)$

$$\iint_S f(x,y,z)dS = \iint_D f(\vec{\Phi}(u,v))\left|\vec{T}_u \times \vec{T}_v\right|dudv$$

Ex. Evaluate $\iint_S (x^2+y^2+z)dS$, where S is given by

$x = rcos\theta, \quad y = rsin\theta, \quad z = r; \quad 0 \le r \le 1; \quad 0 \le \theta \le 2\pi.$

b. Surface integral of a vector field $\vec{F}(x,y,z)$

$\iint_S \vec{F}\cdot d\vec{S} = \iint_D \vec{F}(\vec{\Phi}(u,v))\cdot(\vec{T}_u \times \vec{T}_v)dudv = \iint_S (\vec{F}\cdot\vec{n})dS; \vec{n}$ =unit normal

Ex. Find $\iint_S < z,y,x > \cdot\, d\vec{S}$, where S is given by:

$\vec{\Phi}(\phi,\theta) =< cos\theta sin\phi, sin\theta sin\phi, cos\phi >; \ 0 \le \phi \le \pi \quad 0 \le \theta \le 2\pi.$

2. For surfaces given by $z = g(x,y)$

a. Surface integral of a scalar function $f(x,y,z)$

$$\iint_S f(x,y,z)dS = \iint_D f(x,y,g(x,y))\sqrt{1+(g_x)^2+(g_y)^2}\, dxdy$$

Ex. Evaluate $\iint_S ydS$, where S is the surface $z = x + y^2, \ 0 \le x \le 1 \quad 0 \le y \le 2.$

b. Surface integral of a vector field $\vec{F}(x,y,z)$

$$\iint_S \vec{F}\cdot d\vec{S} = \iint_D (-(F_1)g_x - (F_2)g_y + F_3)\, dxdy$$

Ex. Find $\iint_S \vec{F}\cdot d\vec{S}$ where $\vec{F} =< y,x,z >$ and S is $g(x,y) = 1 - x^2 - y^2, z \ge 0.$