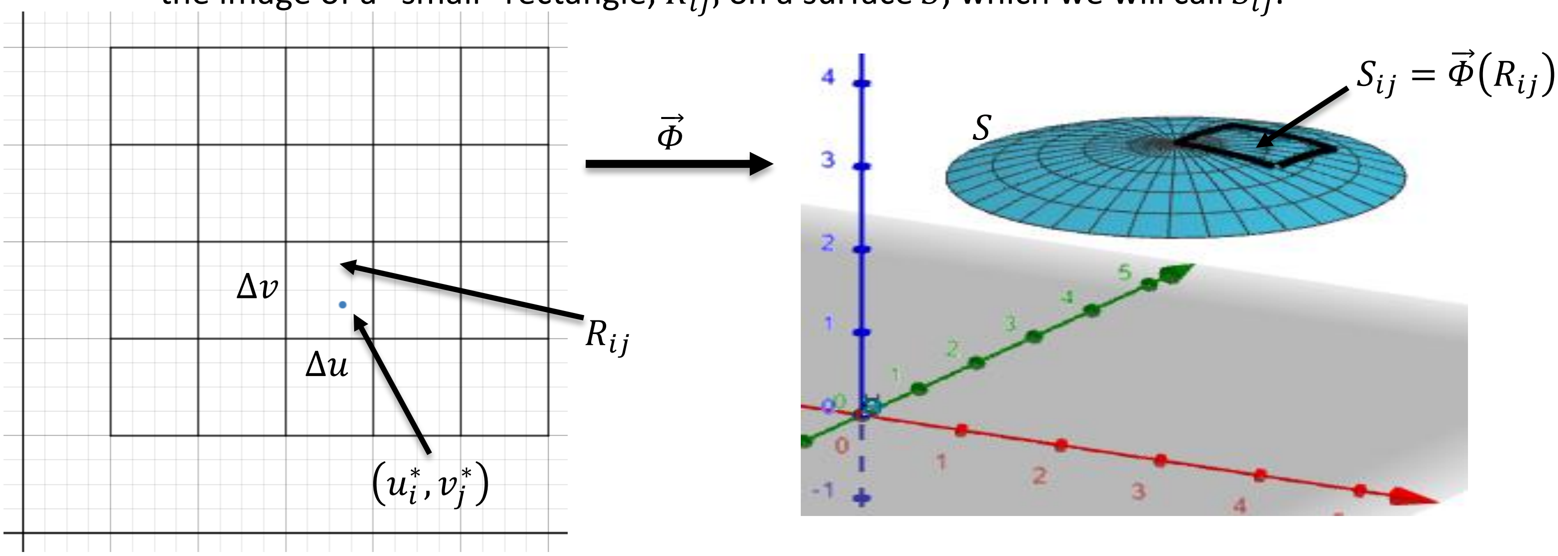
Surface Area of Parametric Surfaces

Def. Let $\vec{\Phi}: D \subset \mathbb{R}^2 \to \mathbb{R}^3$, be a parametrization of a C^1 surface S. We define the surface area of S to be:

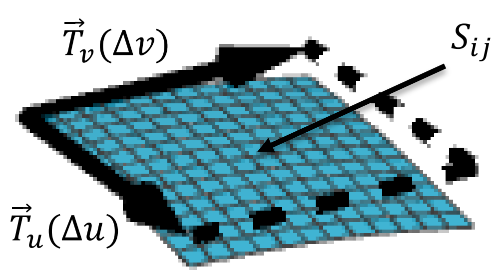
$$\boldsymbol{A(S) = \iint_D \left|\vec{T}_u \times \vec{T}_v\right| dudv}$$

where $\vec{T}_u = \frac{\partial \vec{\Phi}}{\partial u}$ and $\vec{T}_v = \frac{\partial \vec{\Phi}}{\partial v}$.

The motivation for this definition comes from approximating the surface area of the image of a "small" rectangle, R_{ij}, on a surface S, which we will call S_{ij}.



Notice that the area of the parallelogram spanned by $\vec{T}_u(\Delta u)$ and $\vec{T}_v(\Delta v)$ is given by $|\vec{T}_u(\Delta u) \times \vec{T}_v(\Delta v)|$ and is approximately equal to the surface area of S_{ij}, $A(S_{ij})$.



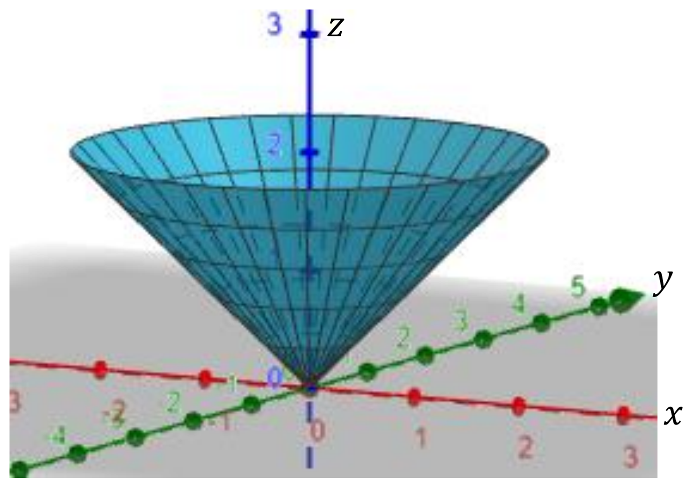
$$A(S) \approx \sum_{i=1}^{n} \sum_{j=1}^{m} |\vec{T}_u(\Delta u) \times \vec{T}_v(\Delta v)| = \sum_{i=1}^{n} \sum_{j=1}^{m} |\vec{T}_u \times \vec{T}_v|(\Delta u)(\Delta v).$$

Now take a limit as $\Delta u, \Delta v$ go to zero to get:

$$A(S) = \iint_D \left|\vec{T}_u \times \vec{T}_v\right| dudv$$

Ex. Find the surface area of the portion of a cone defined by:

$x = rcos\theta \qquad y = rsin\theta \qquad z = r\,; \qquad 0 \le r \le 2, \quad 0 \le \theta \le 2\pi.$



$$\vec{\Phi}(r,\theta) =< rcos\theta, rsin\theta, r >$$

$$\vec{T}_r =< cos\theta, sin\theta, 0 >$$

$$\vec{T}_\theta =< -rsin\theta, rcos\theta, 1 >$$

$$\vec{T}_r \times \vec{T}_\theta = \begin{vmatrix} \vec{\imath} & \vec{\jmath} & \vec{k} \\ cos\theta & sin\theta & 1 \\ -rsin\theta & rcos\theta & 0 \end{vmatrix}$$

$= -rcos\theta\vec{\imath} - rsin\theta\vec{\jmath} + r\vec{k}\,;$ So we have:

$$\left|\vec{T}_r \times \vec{T}_\theta\right| = \sqrt{r^2cos^2\theta + r^2sin^2\theta + r^2}\,.$$

$= |r|\sqrt{2}$ ($r > 0$, so don't need | |).

$$A(S) = \iint_D \left|\vec{T}_u \times \vec{T}_v\right| dudv = \int_{\theta=0}^{\theta=2\pi} \int_{r=0}^{r=2} r\sqrt{2}\, drd\theta$$

$$A(S) = \sqrt{2} \int_{\theta=0}^{\theta=2\pi} \int_{r=0}^{r=2} r\, drd\theta$$

$$= \sqrt{2} \int_{\theta=0}^{\theta=2\pi} \frac{r^2}{2}\Big|_0^2 d\theta = 2\sqrt{2} \int_{\theta=0}^{\theta=2\pi} d\theta = 4\pi\sqrt{2}.$$

Ex. Find the surface area of $x^2 + y^2 + z^2 = 16$, a sphere of radius 4.

Parametrize the sphere with spherical coordinates:

$$x = 4cos\theta sin\phi \quad y = 4sin\theta sin\phi \quad z = 4cos\phi;$$
$$0 \le \phi \le \pi, \quad 0 \le \theta \le 2\pi$$

$$\vec{\Phi}(\phi, \theta) =< 4cos\theta sin\phi, 4sin\theta sin\phi, 4cos\phi >;$$
$$0 \le \phi \le \pi, \quad 0 \le \theta \le 2\pi.$$

$$\vec{T}_\phi =< 4cos\theta cos\phi, 4sin\theta cos\phi, -4sin\phi >$$

$$\vec{T}_\theta =< -4sin\theta sin\phi, 4cos\theta sin\phi, 0 > .$$

$$\vec{T}_\phi \times \vec{T}_\theta = \begin{vmatrix} \vec{\imath} & \vec{\jmath} & \vec{k} \\ 4cos\theta cos\phi & 4sin\theta cos\phi & -4sin\phi \\ -4sin\theta sin\phi & 4cos\theta sin\phi & 0 \end{vmatrix}$$
$$= (16cos\theta \sin^2\phi)\vec{\imath} + (16sin\theta \sin^2\phi)\vec{\jmath} + (16sin\,\phi cos\phi)\vec{k}.$$

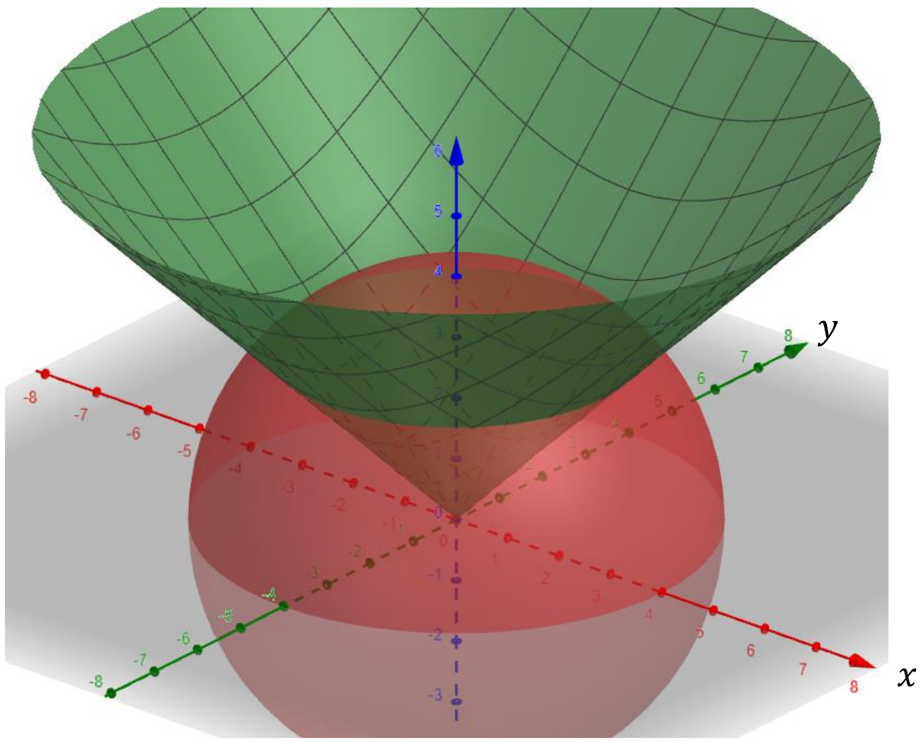
$$\left|\vec{T}_\phi \times \vec{T}_\theta\right| = \sqrt{16^2[\cos^2\theta \sin^4\phi + \sin^2\theta \sin^4\phi + \sin^2\phi \cos^2\phi]}$$
$$= 16\sqrt{\sin^4\phi(\cos^2\theta + \sin^2\theta) + \sin^2\phi \cos^2\phi}$$
$$= 16\sqrt{\sin^2\phi(\sin^2\phi + \cos^2\phi)} = 16sin\phi.$$

$$A(S) = \iint_D \left|\vec{T}_u \times \vec{T}_v\right| dudv = \int_{\theta=0}^{\theta=2\pi} \int_{\phi=0}^{\phi=\pi} 16(\sin\phi)\, d\phi d\theta$$

$$= \int_{\theta=0}^{\theta=2\pi} -16\cos\phi\Big|_0^{\pi} d\theta = \int_{\theta=0}^{\theta=2\pi} 32 d\theta = 64\pi.$$

Ex. Find the surface area of the portion of $x^2 + y^2 + z^2 = 16$ cut out by the cone $z^2 = x^2 + y^2$, $z \geq 0$, with $z^2 \geq x^2 + y^2$.

First draw a picture:



Now find the intersection of the sphere $x^2 + y^2 + z^2 = 16$ and the cone $z^2 = x^2 + y^2$ by solving the equations simultaneously.

Substituting $x^2 + y^2 = z^2$ into $x^2 + y^2 + z^2 = 16$ we get

$z^2 + z^2 = 16$ or $z = \pm 2\sqrt{2}$.

Since $z \geq 0$ in this problem, $z = 2\sqrt{2}$.

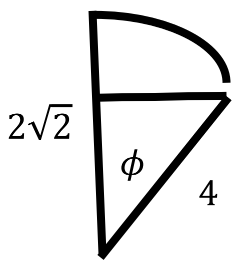
Now plugging $z = 2\sqrt{2}$ into $z^2 = x^2 + y^2$, we get $8 = x^2 + y^2$.

Thus the intersection of this sphere and cone is the circle $x^2 + y^2 = 8$ in the plane $z = 2\sqrt{2}$.

We are finding the surface area of a portion of $x^2 + y^2 + z^2 = 16$ (instead of the entire surface area as we did in the last example). So we already know what

$\left|\vec{T}_\phi \times \vec{T}_\theta\right|$ equals from the last problem. The only difference here is the limits of integration. For the portion of the sphere that we are interested in, what are the limits on ϕ and θ?

The surface area we are finding is the top of the ice cream cone. Thus it's the region where θ goes from 0 to 2π and ϕ, the angle with the z axis, goes from 0, i.e. the north pole, to the point where $z = 2\sqrt{2}$. Thus we can form a triangle:



So $cos\phi = \frac{2\sqrt{2}}{4} = \frac{\sqrt{2}}{2}$; so $\phi = \frac{\pi}{4}$.

$$A(S) = \iint_D \left|\vec{T}_u \times \vec{T}_v\right| dudv = \int_{\theta=0}^{\theta=2\pi} \int_{\phi=0}^{\phi=\frac{\pi}{4}} (16sin\phi)\, d\phi d\theta$$

$$= 16\left(\int_{\theta=0}^{\theta=2\pi} d\theta\right)\left(\int_{\phi=0}^{\phi-\frac{\pi}{4}} (sin\phi) d\phi\right.$$

$$= 16\left(\theta\Big|_0^{2\pi}\right)(-cos\phi)\Big|_0^{\frac{\pi}{4}}$$

$$= 16(2\pi)\left(-\frac{\sqrt{2}}{2} + 1\right) = 32\pi\left(1 - \frac{\sqrt{2}}{2}\right).$$

<u>Surface Area when the Surface is of the form $z = f(x, y)$</u>

If $z = f(x, y)$, we can always parametrize the surface by

$$x = u, \quad y = v, \quad z = f(u, v), \text{ ie } \vec{\Phi}(u, v) =< u, v, f(u, v) >.$$

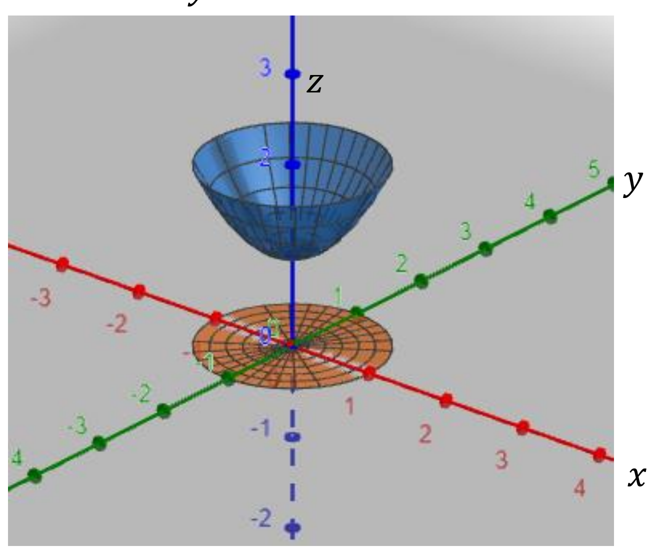
$$\vec{T}_u =< 1, 0, f_u > \qquad \vec{T}_v =< 0, 1, f_v >$$

$$\vec{T}_u \times \vec{T}_v = \begin{vmatrix} \vec{\imath} & \vec{\jmath} & \vec{k} \\ 1 & 0 & f_u \\ 0 & 1 & f_v \end{vmatrix} = -f_u\vec{\imath} - f_v\vec{\jmath} + \vec{k}$$

$$\left|\vec{T}_u \times \vec{T}_v\right| = \sqrt{1 + (f_u)^2 + (f_v)^2}$$

$$\boldsymbol{A(S) = \iint_D \sqrt{1 + (f_u)^2 + (f_v)^2}\, dudv = \iint_D \sqrt{1 + (f_x)^2 + (f_y)^2}\, dxdy.}$$

Ex. Find the surface area of the paraboloid $z = 1 + x^2 + y^2$ that lies above the disk in the xy-plane $x^2 + y^2 \le 1$.



$$f_x = z_x = 2x$$

$$f_y = z_y = 2y$$

$$A(S) = \iint_D \sqrt{1 + (f_x)^2 + (f_y)^2}\, dxdy$$

$$= \iint_{x^2+y^2 \le 1} \sqrt{1 + 4x^2 + 4y^2}\, dxdy$$

Now change to polar coordinates:

$$A(S) = \int_{\theta=0}^{2\pi}\int_{r=0}^{r=1}(\sqrt{1+4r^2})\,r\,dr\,d\theta$$

Let $u = 1 + 4r^2$, when $r = 0$, $u = 1$

$du = 8r\,dr$, when $r = 1$, $u = 5$.

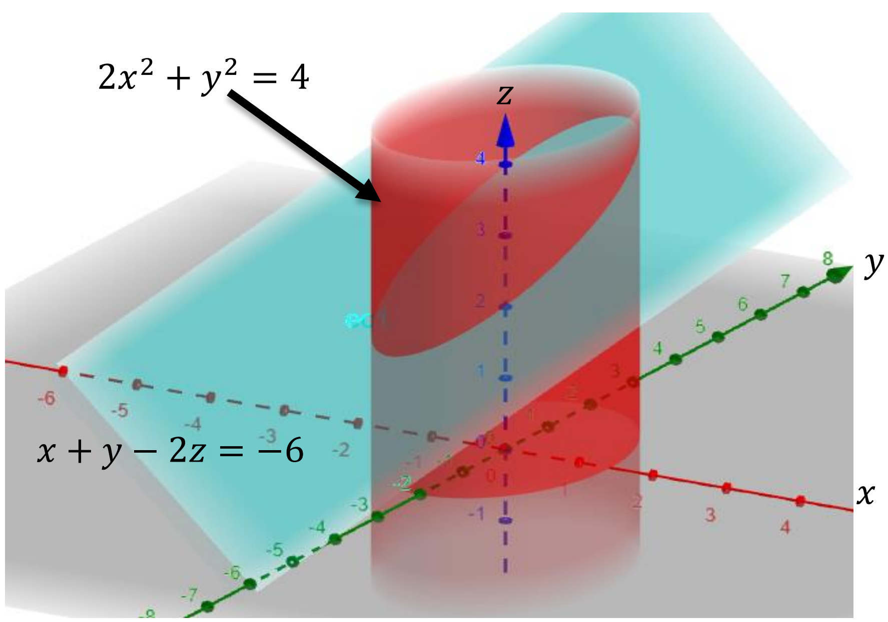
$\frac{1}{8}du = r\,dr$

$$A(S) = \int_{\theta=0}^{\theta=2\pi}\int_{u=1}^{u=5} u^{\frac{1}{2}}\left(\frac{1}{8}\right)du\,d\theta$$

$$= \left(\int_{\theta=0}^{\theta=2\pi} d\theta\right)\left(\int_{u=1}^{u=5} u^{\frac{1}{2}}\left(\frac{1}{8}\right)du\right)$$

$$= (2\pi)\left(\frac{1}{8}\right)\left(\frac{2}{3}u^{\frac{3}{2}}\Big|_1^5 = \frac{\pi}{6}(5\sqrt{5}-1).$$

Ex. Find the surface area of the portion of the plane $x + y - 2z = -6$ where $2x^2 + y^2 \le 4$.



$z = \frac{1}{2}x + \frac{1}{2}y + 3$;   So we can write the surface as $z = f(x, y)$.

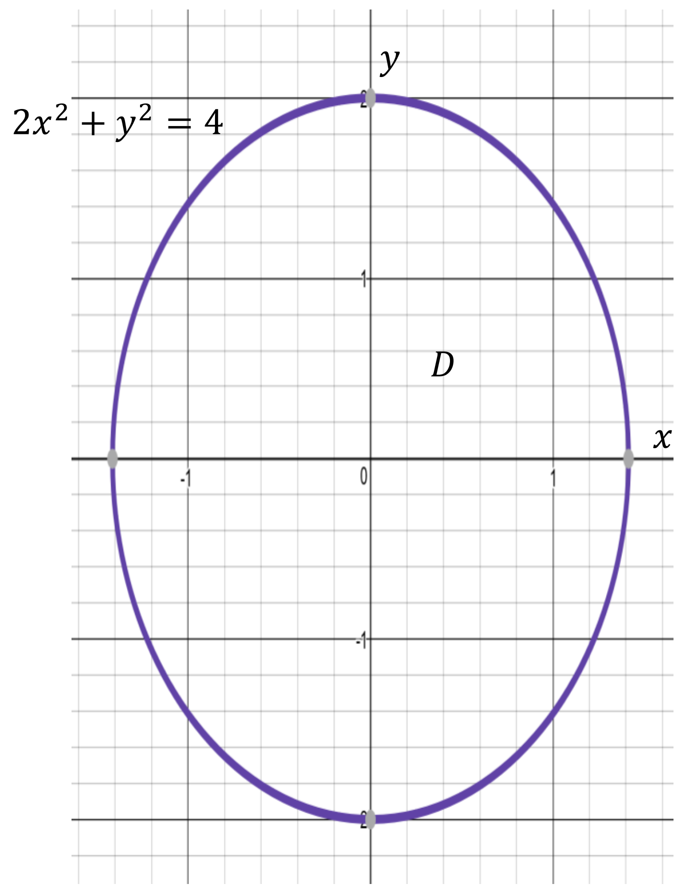
$z_x = f_x = \frac{1}{2}$

$z_y = f_y = \frac{1}{2}$

$$A(S) = \iint_D \sqrt{1 + (f_x)^2 + (f_y)^2}\, dxdy$$

$$A(S) = \iint_{2x^2+y^2\leq 4} \sqrt{1 + (\tfrac{1}{2})^2 + (\tfrac{1}{2})^2}\, dxdy = \iint_{2x^2+y^2\leq 4} \sqrt{\tfrac{3}{2}}\, dxdy$$

The region D is the interior of the ellipse $2x^2 + y^2 = 4$. If we let:



$u = \sqrt{2}x$

$du = \sqrt{2}dx$

$\frac{1}{\sqrt{2}}du = dx$

The region we will now be integrating over will be a disk $u^2 + y^2 \leq 4$.

$$A(S) = \frac{1}{\sqrt{2}} \iint_{u^2+y^2\leq 4} \sqrt{\tfrac{3}{2}}\, dudy\ .$$

Now change to polar coordinates:

$u = rcos\theta$

$y = rsin\theta$

$dudy = rdrd\theta$

$$A(S) = \frac{\sqrt{3}}{2} \int_{\theta=0}^{\theta=2\pi} \int_{r=0}^{r=2} rdrd\theta = \frac{\sqrt{3}}{2} \left(\int_{\theta=0}^{\theta=2\pi} d\theta\right)\left(\int_{r=0}^{r=2} rdr\right) = 2\pi\sqrt{3}.$$

Surface Area for Surfaces of Revolution

From first year Calculus we know that if the curve $y = f(x)$ is revolved about the x-axis, the surface area is given by: $A(S) = 2\pi \int_{x=a}^{x=b} |f(x)|\sqrt{1 + (f'(x))^2}\, dx$.

We will now rederive this formula using the surface area formula for parametric surfaces.

We can parametrize the surface of revolution produced by revolving $y = f(x)$ about the x-axis by:

$$x = u \quad y = f(u)cosv \quad z = f(u)sinv; \quad a \le u \le b, \quad 0 \le v \le 2\pi.$$

Thus: $\vec{\Phi}(u, v) =< u, f(u)cosv, f(u)sinv >; \ a \le u \le b, \quad 0 \le v \le 2\pi.$

$$\vec{T}_u =< 1, f'(u)cosv, f'(u)sinv >$$
$$\vec{T}_v =< 0, -f(u)sinv, f(u)cosv >$$

$$\vec{T}_u \times \vec{T}_v = \begin{vmatrix} \vec{\imath} & \vec{\jmath} & \vec{k} \\ 1 & f'(u)cosv & f'(u)sinv \\ 0 & -f(u)sinv & f(u)cosv \end{vmatrix}$$

$$\vec{T}_u \times \vec{T}_v = (f'(u)f(u)cos^2v + f(u)f'(u)sin^2v)\vec{\imath} - (f(u)cosv)\vec{\jmath} - (f(u)sinv)\vec{k}$$

$$\vec{T}_u \times \vec{T}_v = (f'(u)f(u))\vec{\imath} - (f(u)cosv)\vec{\jmath} - (f(u)sinv)\vec{k}$$

$$\left|\vec{T}_u \times \vec{T}_v\right| = \sqrt{(f'(u)f(u))^2 + (f(u))^2cos^2v + (f(u))^2sin^2v}$$

$$\left|\vec{T}_u \times \vec{T}_v\right| = \sqrt{(f'(u)f(u))^2 + (f(u))^2} = |f(u)|\sqrt{1 + (f'(u))^2}$$

$$A(S) = \int_{v=0}^{2\pi} \int_{u=a}^{b} |f(u)| \sqrt{1 + (f'(u))^2}\ \mathrm{dudv}$$

$$A(S) = \int_{v=0}^{2\pi} dv \int_{u=a}^{b} |f(u)| \sqrt{1 + (f'(u))^2}\ du$$

$$A(S) = 2\pi \int_{u=a}^{b} |f(u)| \sqrt{1 + (f'(u))^2}\ \mathrm{du}\ .$$