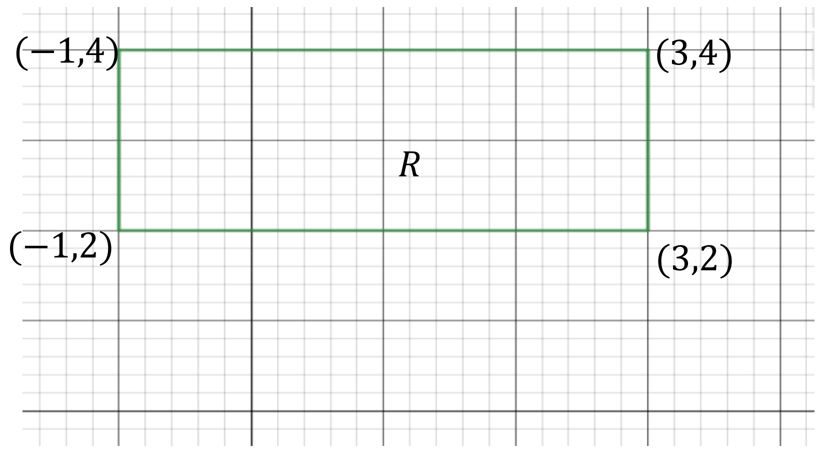
A Quick Review of Multiple Integration

Double Integration

If the x and y endpoints of integration are all constants (i.e., just numbers, no variables), then you are integrating over a rectangular region in the plane whose sides are parallel to the x and y axes.

Ex. $\int_{y=2}^{y=4}\int_{x=-1}^{x=3} 2xy^2dxdy$, is an integral over the rectangular region R

$-1 \leq x \leq 3$ and $2 \leq y \leq 4$



To evaluate this integrate, we first integrate with respect to x, holding y constant, substitute the values of x, and then integrate with respect to y.

$$\int_{y=2}^{y=4}\int_{x=-1}^{x=3} 2xy^2dxdy = \int_{y=2}^{y=4} x^2y^2 \,|_{x=-1}^{x=3} dy = \int_{y=2}^{y=4} y^2(3^2-(-1)^2)dy$$

$$= \int_{y=2}^{y=4} 8y^2\, dy = \frac{8}{3}y^3|_{y=2}^{y=4} = \frac{8}{3}(4^3-2^3) = \frac{448}{3}.$$

When the endpoints of integration are all constants and the function you are integrating can be written as $f(x,y) = g(x)h(y)$, then

$$\int_{y=c}^{y=d}\int_{x=a}^{x=b} \boldsymbol{f(x,y)dxdy} = \int_{y=c}^{y=d}\int_{x=a}^{x=b} \boldsymbol{g(x)h(y)dxdy}\,.$$

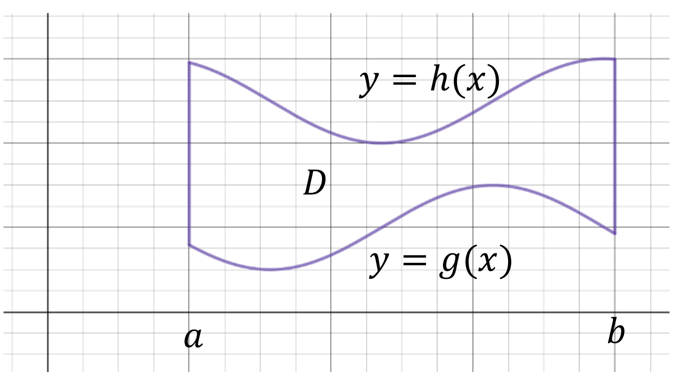
$$= \left(\int_{y=c}^{y=d} \boldsymbol{h(y)dy}\right)\left(\int_{x=a}^{x=b} \boldsymbol{g(x)dx}\right).$$

Ex. $\int_{y=2}^{y=4}\int_{x=-1}^{x=3} 2xy^2dxdy = \left(\int_{y=2}^{y=4} y^2dy\right)\left(\int_{x=-1}^{x=3} 2xdx\right)$

$$= \left(\frac{1}{3}y^3\Big|_{y=2}^{y=4}\right)\left(x^2\Big|_{x=-1}^{x=3}\right) = \frac{1}{3}(64-8)(9-1) = \frac{448}{3}\,.$$

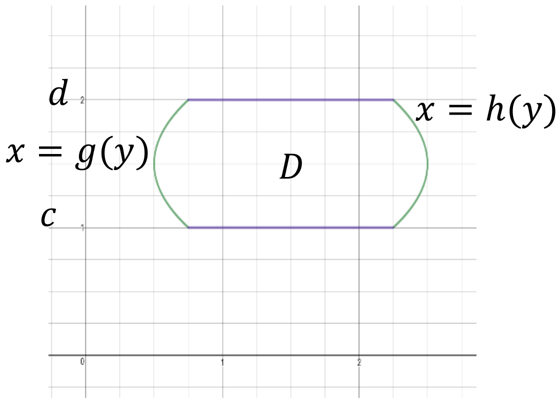
When the region we are integrating over is not a rectangle, the endpoints in x and y will not all be constant. We have 2 methods we can consider in this case.

1. If the region D is bounded below by $y = g(x)$ and above by $y = h(x)$, and along the sides by $x = a$ and $x = b$, we have:



$$\iint_D f(x,y)dA = \int_{x=a}^{x=b}\int_{y=g(x)}^{y=h(x)} f(x,y)dydx$$

2. If the region D is bounded on the left side by $x = g(y)$ and on the right side by $x = h(y)$, and below by $y = c$ and above by $y = d$, then we have:



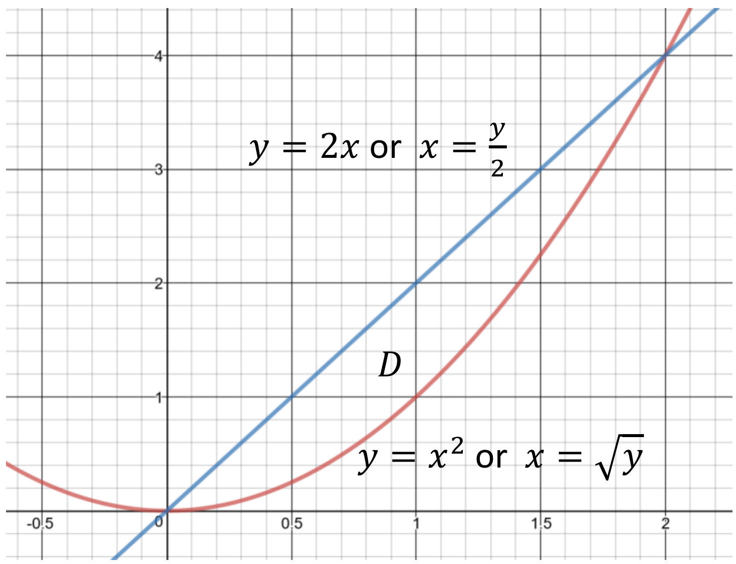
$$\iint_D f(x,y)dA = \int_{y=c}^{y=d} \int_{x=g(y)}^{x=h(y)} f(x,y)dxdy.$$

Note: **If $\boldsymbol{f(x,y) = 1}$, then $\boldsymbol{\iint_D f(x,y)dA = \iint_D 1\, dA}$ =Area of the region $\boldsymbol{D}$.**

Ex. Evaluate $\iint_D x^2 y dA$, where D is the region bounded by the curves $y = x^2$ and $y = 2x$.

In this case we can solve this using either method 1 or method 2.

Start by graphing the curves that bound D. Find the points of intersection of the curves (in this case setting $x^2 = 2x$ we find $x = 0$ and $x = 2$). The points of intersection are $(0,0)$ and $(2,4)$.



Method 1:

The bottom curve is $y = x^2$ and the top curve is $y = 2x$.

$$\iint_D x^2 y dA = \int_{x=0}^{x=2} \int_{y=x^2}^{y=2x} x^2 y dy dx$$

$$= \int_{x=0}^{x=2} \frac{x^2}{2} y^2 \Big|_{y=x^2}^{y=2x} dx$$

$$= \int_{x=0}^{x=2} \frac{x^2}{2} ((2x)^2 - (x^2)^2) dx$$

$$= \int_{x=0}^{x=2} (2x^4 - \tfrac{1}{2} x^6)\, dx = \left(\tfrac{2}{5} x^5 - \tfrac{1}{14} x^7\right) \Big|_{x=0}^{x=2} = \frac{128}{35}.$$

Method 2:

The left curve is $x = \frac{y}{2}$ and the right curve is $x = \sqrt{y}$.

$$\iint_D x^2 y dA = \int_{y=0}^{y=4} \int_{x=\frac{y}{2}}^{x=\sqrt{y}} x^2 y dx dy = \int_{y=0}^{y=4} \frac{x^3}{3} y |_{x=\frac{y}{2}}^{x=\sqrt{y}} dy$$

$$= \int_{y=0}^{y=4} [\frac{y^{\frac{3}{2}}}{3}(y) - \left(\frac{y^3}{24}\right) y] dy$$

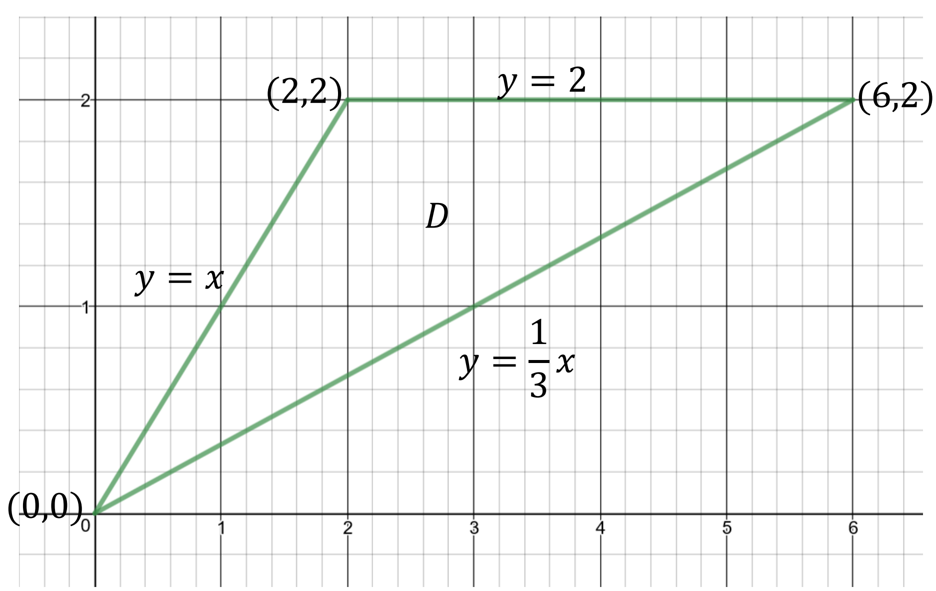
$$= \int_{y=0}^{y=4} (\frac{1}{3} y^{\frac{5}{2}} - \frac{1}{24} y^4)\, dy$$

$$= \left(\left(\frac{1}{3}\right)\left(\frac{2}{7}\right) y^{\frac{7}{2}} - \frac{1}{120} y^5 \right) |_{y=0}^{y=4} = \frac{128}{35}.$$

Ex. Evaluate $\iint_D 2xy dA$ where D is the region inside the triangle with vertices $(0,0), (2,2), (6,2)$.

First find the equations of the 3 sides:

$y = \frac{1}{3}x, \quad y = x,$ and $y = 2.$



Method 1.

Notice that the top curve switches at $x = 2$ (from $y = x$ to $y = 2$). Thus we have to break the integral up into 2 pieces if we want to use this method.

$$\iint_D 2xydA = \int_{x=0}^{x=2} \int_{y=\frac{1}{3}x}^{y=x} 2xydydx + \int_{x=2}^{x=6} \int_{y=\frac{1}{3}x}^{y=2} 2xydydx$$

$$= \int_{x=0}^{x=2} xy^2 \Big|_{y=\frac{1}{3}x}^{y=x} dx + \int_{x=2}^{x=6} xy^2 \Big|_{y=\frac{1}{3}x}^{y=2} dy$$

$$= \int_{x=0}^{x=2} \left(x^3 - \frac{x^3}{9}\right) dx + \int_{x=2}^{x=6} \left(4x - \frac{x^3}{9}\right) dx$$

$$= \int_{x=0}^{x=2} \frac{8}{9} x^3 dx + \int_{x=2}^{x=6} \left(4x - \frac{x^3}{9}\right) dx$$

$$= \frac{2}{9} x^4 \Big|_{x=0}^{x=2} + \left(2x^2 - \frac{1}{36} x^4\right) \Big|_{x=2}^{x=6} = 32 \,.$$

Method 2 (the easier way for this problem).

The left curve is $x = y$ and the right curve is $x = 3y$.

$$\iint_R 2xydA = \int_{y=0}^{y=2} \int_{x=y}^{x=3y} 2xydxdy.$$

$$= \int_{y=0}^{y=2} x^2 y \Big|_{x=y}^{x=3y} dy$$

$$= \int_{y=0}^{y=2} (9y^3 - y^3) dy = \int_{y=0}^{y=2} 8y^3 dy = 2y^4 \Big|_{y=0}^{y=2} = 32 \,.$$

When integrating over a disk, or an annulus, or a portion of a disk or an annulus, it is often useful to change to polar coordinates:

$x = r\cos\theta$

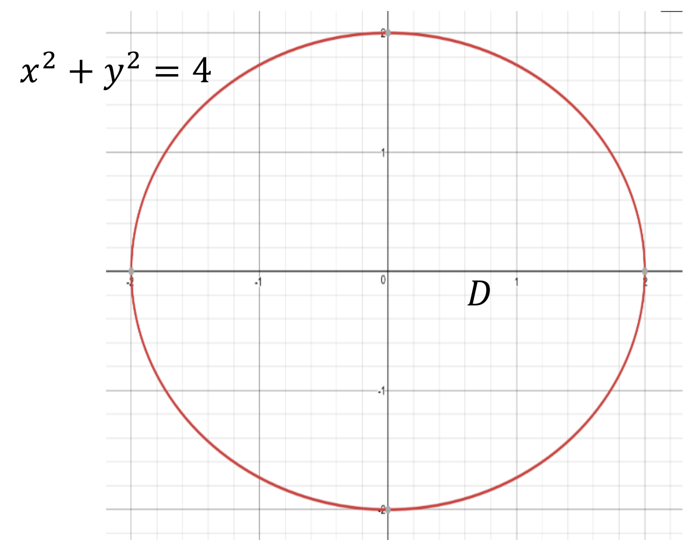
$y = r\sin\theta$

$x^2 + y^2 = r^2$

$dA = r\,dr\,d\theta$

Ex. Evaluate $\iint_D (x^2+y^2)^2\,dy\,dx$; where $D = \{(x,y)|\ x^2+y^2 \le 4\}$

First sketch the region D:



$$\iint_D (x^2+y^2)^2\,dy\,dx = \int_{\theta=0}^{\theta=2\pi}\int_{r=0}^{r=2}(r^2)^2\,r\,dr\,d\theta$$

$$= \int_{\theta=0}^{\theta=2\pi}\int_{r=0}^{r=2} r^5\,dr\,d\theta$$

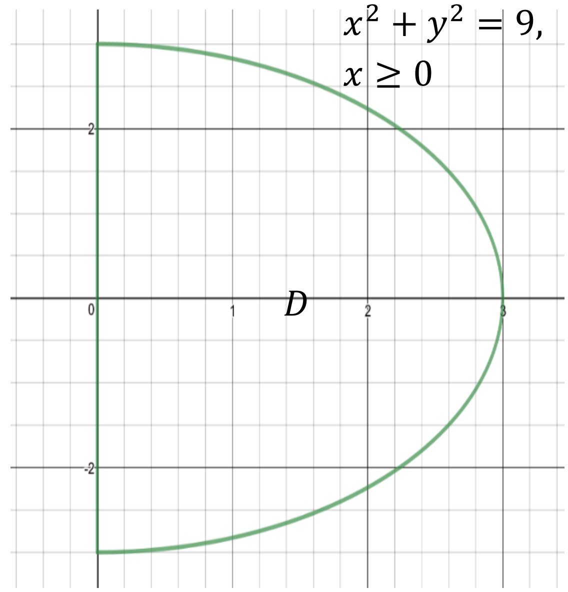
$$= \int_{\theta=0}^{\theta=2\pi} \frac{r^6}{6}\Big|_{r=0}^{r=2}\,d\theta$$

$$= \int_{\theta=0}^{\theta=2\pi} \frac{1}{6}(2^6 - 0^6)\,d\theta$$

$$= \int_{\theta=0}^{\theta=2\pi} \frac{32}{3}\,d\theta = \frac{32}{3}\theta\Big|_0^{2\pi} = \frac{64\pi}{3}.$$

Ex. Evaluate $\iint_D xydA$, where D is the set where $x^2+y^2 \le 9$ and $x \ge 0$.

First sketch the region D:



$$\iint_D xydA = \int_{\theta=-\frac{\pi}{2}}^{\theta=\frac{\pi}{2}} \int_{r=0}^{r=3} (rcos\theta)(rsin\theta)rdrd\theta$$

$$= \int_{\theta=-\frac{\pi}{2}}^{\theta=\frac{\pi}{2}} \int_{r=0}^{r=3} r^3 cos\theta sin\theta drd\theta$$

Since the endpoints of integration are constants and $f(r,\theta) = g(r)h(\theta)$ we have

$$= \int_{\theta=-\frac{\pi}{2}}^{\theta=\frac{\pi}{2}} cos\theta sin\theta d\theta \int_{r=0}^{r=3} r^3\, dr$$

Let $u = sin\theta$; when $\theta = -\frac{\pi}{2}$, $u=-1$

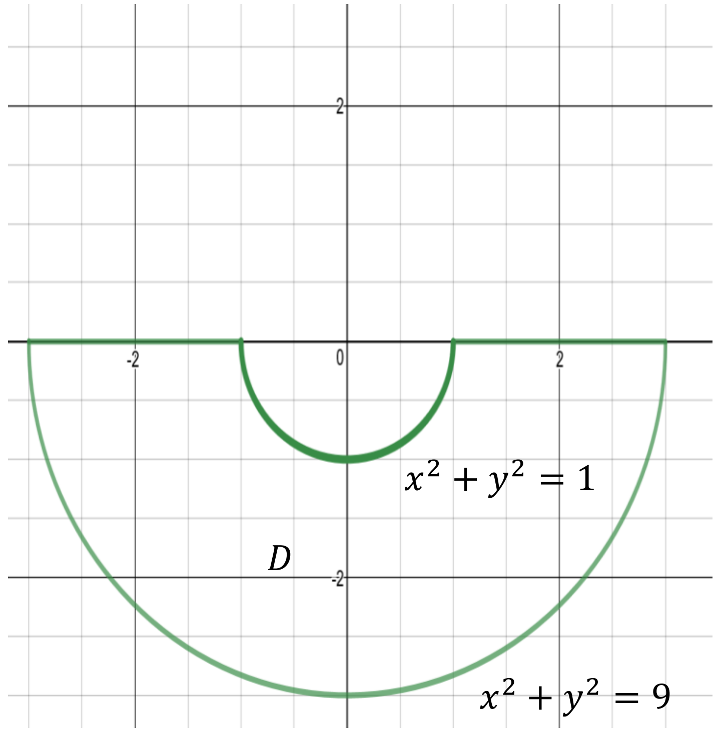
$du = cos\theta d\theta$ when $\theta = \frac{\pi}{2}$, $u = 1$

$$\iint_D xydA = (\int_{-1}^{1} udu)\,(\tfrac{1}{4} r^4|_{r=0}^{r=4})$$

$$= \left(\tfrac{1}{2}u^2|_{u=-1}^{u=1}\right)\left(\tfrac{1}{4}(4^4)\right) = (0)(4^3) = 0.$$

Ex. Evaluate $\iint_D x^2 dydx$, where D=$\{(x,y)|\ 1 \le x^2+y^2 \le 9,\ and\ \ y \le 0\}$

First sketch the region D.



$$\iint_D x^2 dydx = \int_{\theta=\pi}^{\theta=2\pi} \int_{r=1}^{r=3} (r^2 \cos^2\theta) r dr d\theta$$

$$= \int_{\theta=\pi}^{\theta=2\pi} \int_{r=1}^{r=3} (r^3 \cos^2\theta) dr d\theta$$

Since the endpoints of integration are constants and $f(r,\theta) = g(r)h(\theta)$ we have

$$\iint_D x^2 dydx = \left(\int_{\theta=\pi}^{\theta=2\pi} \cos^2\theta d\theta\right)\left(\int_{r=1}^{r=3} r^3 dr\right)$$

$$= \left(\int_{\theta=\pi}^{\theta=2\pi} \left(\frac{1}{2}+\frac{1}{2}cos2\theta\right) d\theta\right)\left(\frac{1}{4}r^4 \Big|_{r=1}^{r=3}\right)$$

$$= \left[\left(\frac{1}{2}\theta + \frac{1}{4}sin2\theta\right) \Big|_{\theta=\pi}^{\theta=2\pi}\right]\left(\frac{1}{4}(3^4-1^4)\right)$$

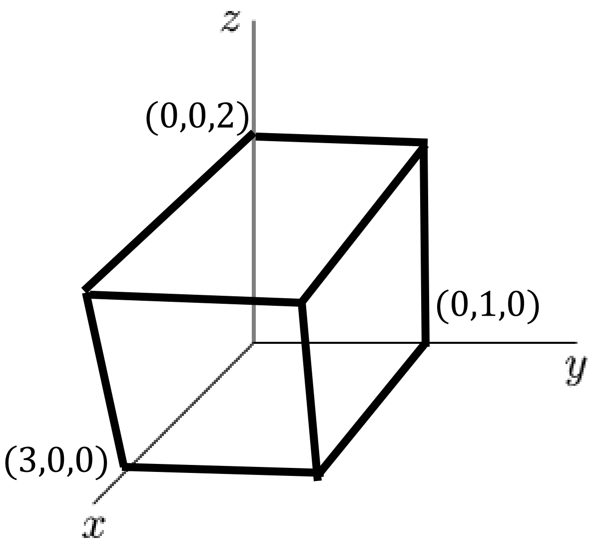
$$= \left[(\pi+0) - \left(\frac{\pi}{2}+0\right)\right]\left(\frac{80}{4}\right) = 10\pi\ .$$

<u>Triple Integrals</u>

If the x, y, and z endpoints of integration are all constants, then you are integrating over a rectangular solid whose sides are parallel to the coordinate planes.

Ex. Evaluate $\iiint_W xy\,dW$, if $W = \{(x, y, z) |\ 0 \leq x \leq 3,\ 0 \leq y \leq 1,\ 0 \leq z \leq 2\}$

First sketch W:



$$\int_{x=0}^{x=3}\int_{y=0}^{y=1}\int_{z=0}^{z=2}(xy)dzdydx = \int_{x=0}^{x=3}\int_{y=0}^{y=1} xyz|_{z=0}^{z=2}dydx$$

$$= \int_{x=0}^{x=3}\int_{y=0}^{y=1} xy(2-0)dydx$$

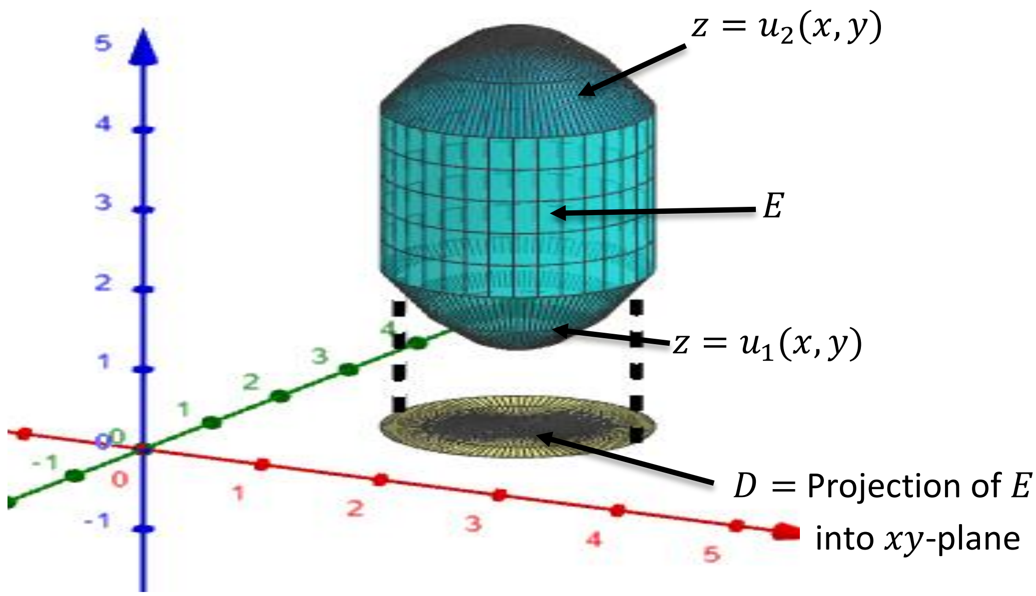
$$= \int_{x=0}^{x=3}\int_{y=0}^{y=1} 2xydydx$$

$$= \int_{x=0}^{x=3} xy^2|_{y=0}^{y=1}dx$$

$$= \int_{x=0}^{x=3} x(1^2-0^2)dx = \int_{x=0}^{x=3} xdx = \frac{9}{2}.$$

If E is bounded above by the surface $z = u_2(x, y)$ and below by the surface $z = u_1(x, y)$, and the projection of the solid E into the xy-plane is D then:

$$\iiint_E f(x,y,z)dE = \iint_D \int_{z=u_1(x,y)}^{z=u_2(x,y)} f(x,y,z)dzdydx$$

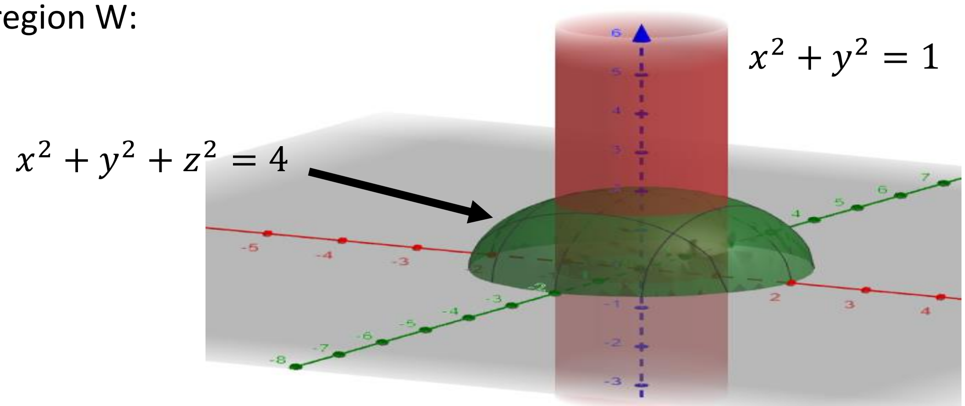


Note: **If $f(x,y,z) = 1$ then $\iiint_E f(x,y,z)dE = \iiint_E 1\, dE =$ Volume of E.**

Ex. Evaluate $\iiint_W x^2 dW$, where

$W = \{(x, y, z)|\ x^2 + y^2 \le 1,\ z \ge 0,\ x^2 + y^2 + z^2 \le 4\}$

First sketch the region W:



$$\iiint_W x^2 dW = \iint_{x^2+y^2\leq 1}\int_{z=0}^{z=\sqrt{4-x^2-y^2}}(x^2)dzdydx$$

$$= \iint_{x^2+y^2\leq 1} x^2 z|_{z=0}^{z=\sqrt{4-x^2-y^2}} dydx$$

$$= \iint_{x^2+y^2\leq 1} x^2(\sqrt{4-x^2-y^2}-0)dydx$$

$$= \iint_{x^2+y^2\leq 1} x^2(\sqrt{4-x^2-y^2})dydx.$$

Now change to polar coordinates since we are integrating over a disk

$$x = rcos\theta, \quad y = rsin\theta, \quad dydx = rdrd\theta$$

$$\iiint_W x^2 dW = \int_{\theta=0}^{\theta=2\pi}\int_{r=0}^{r=1}(r^2\cos^2\theta)\left(\sqrt{4-r^2\cos^2\theta-r^2\sin^2\theta}\right)rdrd\theta$$

$$= \int_{\theta=0}^{\theta=2\pi}\int_{r=0}^{r=1}(r^2\cos^2\theta)\left(\sqrt{4-r^2}\right)rdrd\theta.$$

Since the endpoints of integration are constants and $f(r,\theta) = g(r)h(\theta)$ we have

$$\iiint_W x^2 dW = (\int_{\theta=0}^{\theta=2\pi}\cos^2\theta d\theta)(\int_{r=0}^{r=1} r^3\sqrt{4-r^2}\, dr)$$

Let’s evaluate each integral separately.

$$\int_{\theta=0}^{\theta=2\pi}\cos^2\theta d\theta = \int_{\theta=0}^{\theta=2\pi}(\frac{1}{2}+\frac{1}{2}\cos 2\theta)d\theta$$

$$= \left(\frac{1}{2}\theta+\frac{1}{4}sin2\theta\right)|_{\theta=0}^{\theta=2\pi} = \pi.$$

$$\int_{r=0}^{r=1} r^3\sqrt{4-r^2}\,dr = \int_{r=0}^{r=1} r^2\sqrt{4-r^2}\,rdrd\theta$$

Let $u = 4 - r^2$ which means that $r^2 = 4 - u$. when $r = 0,\quad u = 4$

$du = -2rdr$ when $r = 1,\ \ u = 3.$

$$-\frac{1}{2}du = rdr$$

$$\int_{r=0}^{r=1} r^2\sqrt{4-r^2}\,rdrd\theta = \int_{u=4}^{u=3}(4-u)u^{\frac{1}{2}}(-\frac{1}{2})du$$

$$= -\frac{1}{2}\int_{u=4}^{u=3}(4u^{\frac{1}{2}} - u^{\frac{3}{2}})du$$

$$= -\frac{1}{2}[\frac{8}{3}u^{\frac{3}{2}} - \frac{2}{5}u^{\frac{5}{2}}]|_{u=4}^{u=3}$$

$$= \frac{64}{15} - \frac{11}{5}\sqrt{3}\ .$$

Thus we have:

$$\iiint_W x^2 dW = (\int_{\theta=0}^{\theta=2\pi}\cos^2\theta d\theta)\left(\int_{r=0}^{r=1} r^3\sqrt{1-r^2}\,dr\right) = (\pi)(\frac{64}{15} - \frac{11}{5}\sqrt{3})\ .$$