Parametrized Surfaces

Just as it's sometimes simpler to represent a curve in $\mathbb{R}^3$ $(or\ \mathbb{R}^2)$ in terms of parametric equations: $x = x(t),\ y = y(t),\ z = z(t)$, instead of trying to represent it as the intersection of 2 surfaces, $z = f(x, y)$ and $z = g(x, y)$ (which can't always be done), it is sometimes simpler to represent a surface in $\mathbb{R}^3$ in parametric form: $x = x(u, v),\ y = y(u, v),\ z = z(u, v)$, instead of $z = f(x, y)$ (which can't always be done).

Ex. Even a simple surface like the cylinder $x^2 + z^2 = 4$, can't be represented as a simple function $z = f(x, y)$ (in this case we would have 2 functions $z = \sqrt{4 - x^2}$ and $z = -\sqrt{4 - x^2}$). However, parametrically we can represent this cylinder by: $x = 2cosu,\quad y = v,\quad z = 2sinu\ ;\quad 0 \le u \le 2\pi,\ v \in \mathbb{R}$ (notice that x, y, and z have to satisfy the original equation: $x^2 + z^2 = 4$).

In general, we can represent a surface in parametric form as:

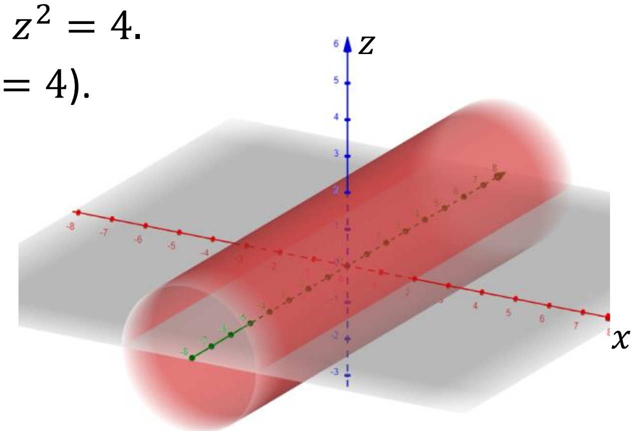
$x = x(u, v),\ y = y(u, v),\ z = z(u, v)$, and in vector form by:

$\vec{\Phi}(u, v) =< x(u, v), y(u, v), z(u, v) >$; where $\vec{\Phi} : D \subset \mathbb{R}^2 \to \mathbb{R}^3$.

The surface, S, is the image of $\vec{\Phi}$, i.e. $\vec{\Phi}(D)$, and $\vec{\Phi}$ is called a **parametrization** of S. For any surface there are an infinite number of parametrizations.

S is called a **Differentiable Surface**, (or a $\boldsymbol{C^1}$ **Surface)** if $x(u, v),\ y(u, v),\ z(u, v)$, are differentiable (or C^1)

Ex. $\vec{\Phi}(u, v) =< 2cosu, v, 2sinu >,\ \ 0 \le u \le 2\pi,\ v \in \mathbb{R}$ is a parametrization of the circular cylinder $x^2 + z^2 = 4$.
(Notice: $x^2 + z^2 = (2cosu)^2 + (2sinu)^2 = 4$).



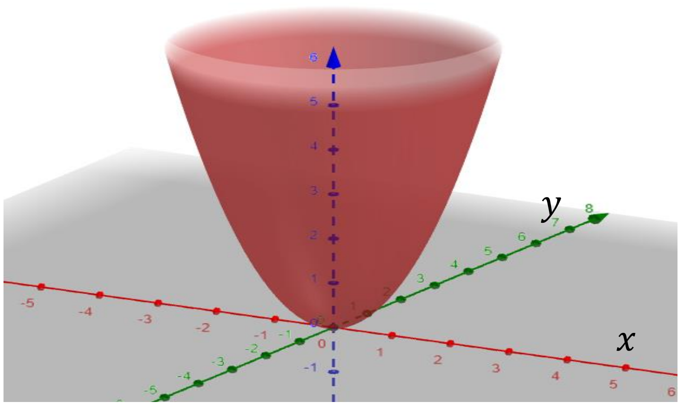
Ex. Notice that any surface $z = f(x, y)$; e.g., $z = x^2 + y^2$, can be parametrized by: $x = u,\ y = v,\ z = f(u, v)$, ie,

$$\vec{\Phi}(u, v) = < u, v, f(u, v) >.$$

In the case of

$$z = x^2 + y^2; \quad x = u, \quad y = v, \quad z = u^2 + v^2, \text{ i.e.,}$$

$$\vec{\Phi}(u, v) = < u, v, u^2 + v^2 >.$$



Ex. (Important Example) Find a parametrization of the sphere of radius R, $x^2 + y^2 + z^2 = R^2$.

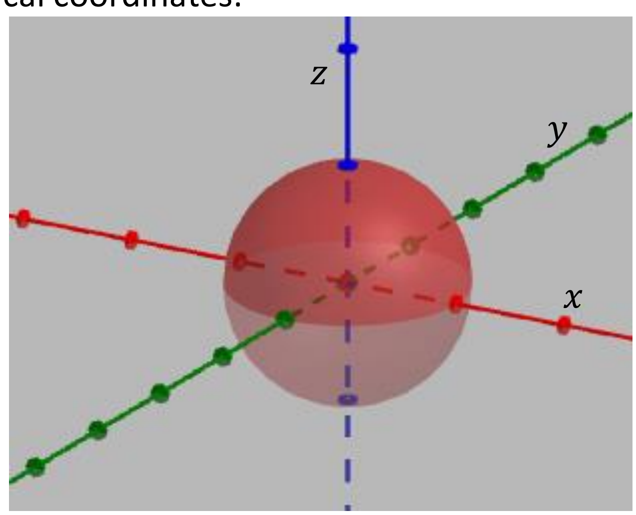
One standard parametrization is to use spherical coordinates:

$$x = R\ cos\theta sin\phi$$

$$y = Rsin\theta sin\phi$$

$$z = Rcos\phi$$

where $0 \le \phi \le \pi$, and $0 \le \theta \le 2\pi$.



Equivalently we could write:

$\vec{\Phi}(\phi,\theta) =< R\ cos\theta sin\phi,\ Rsin\theta sin\phi,\ Rcos\phi >$;

$0 \le \phi \le \pi$, and $0 \le \theta \le 2\pi$.

Notice: $x^2 + y^2 + z^2 = (R\ cos\theta sin\phi)^2 + (Rsin\theta sin\phi)^2 + (Rcos\phi\ )^2$

$$= R^2(sin\phi)^2(cos^2\theta + sin^2\theta) + (Rcos\phi\ )^2$$

$$= R^2(sin^2\phi) + R^2(cos^2\phi) = R^2.$$

Ex. Find a parametrization of the ellipsoid $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$.

Again, using spherical coordinates we get:

$x = a(cos\theta sin\phi)$

$y = b(sin\theta sin\phi)$

$z = c(cos\phi)$

Where $0 \le \phi \le \pi$, and $0 \le \theta \le 2\pi$.

Equivalently we could write:

$\vec{\Phi}(\phi,\theta) =< acos\theta sin\phi,\ bsin\theta sin\phi,\ ccos\phi >$;

$0 \le \phi \le \pi$, and $0 \le \theta \le 2\pi$.

Notice that:

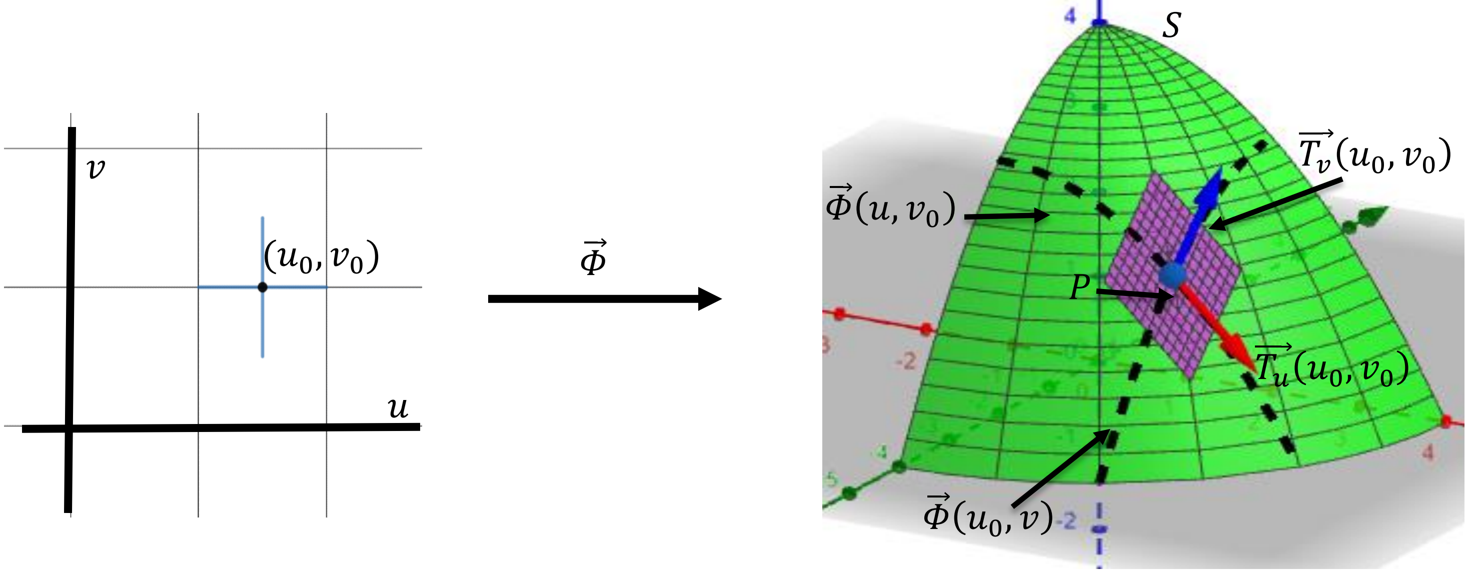
$$(\frac{x}{a})^2 + (\frac{y}{b})^2 + (\frac{z}{c})^2 = cos^2\theta sin^2\phi + sin^2\theta sin^2\phi + cos^2\phi = 1.$$

Tangent Planes to Parametrized Surfaces

Let $\vec{\Phi}$ be a differentiable parametrization of a surface S,

$\vec{\Phi}(u, v) =< x(u, v), y(u, v), z(u, v) >;$ with $\vec{\Phi}(u_0, v_0) = P,$

$\vec{T}_u = \frac{\partial \vec{\Phi}}{\partial u} = < \frac{\partial x}{\partial u}, \frac{\partial y}{\partial u}, \frac{\partial z}{\partial u} >, \quad \vec{T}_v = \frac{\partial \vec{\Phi}}{\partial v} = < \frac{\partial x}{\partial v}, \frac{\partial y}{\partial v}, \frac{\partial z}{\partial v} >.$



At a fixed point (u_0, v_0), the vectors $\vec{T}_u(u_0, v_0)$ and $\vec{T}_v(u_0, v_0)$ are tangent to the surface S at $\vec{\Phi}(u_0, v_0)$. If $\vec{T}_u(u_0, v_0) \times \vec{T}_v(u_0, v_0) \neq 0$, then the surface is called **Regular**, or **Smooth**, at $\vec{\Phi}(u_0, v_0)$. The surface S is called Regular or Smooth if $\vec{T}_u(u, v) \times \vec{T}_v(u, v) \neq 0$ for all points $\vec{\Phi}(u, v) \in S$.

If $\vec{T}_u(u_0, v_0) \times \vec{T}_v(u_0, v_0) \neq 0$ at a fixed point (u_0, v_0), then

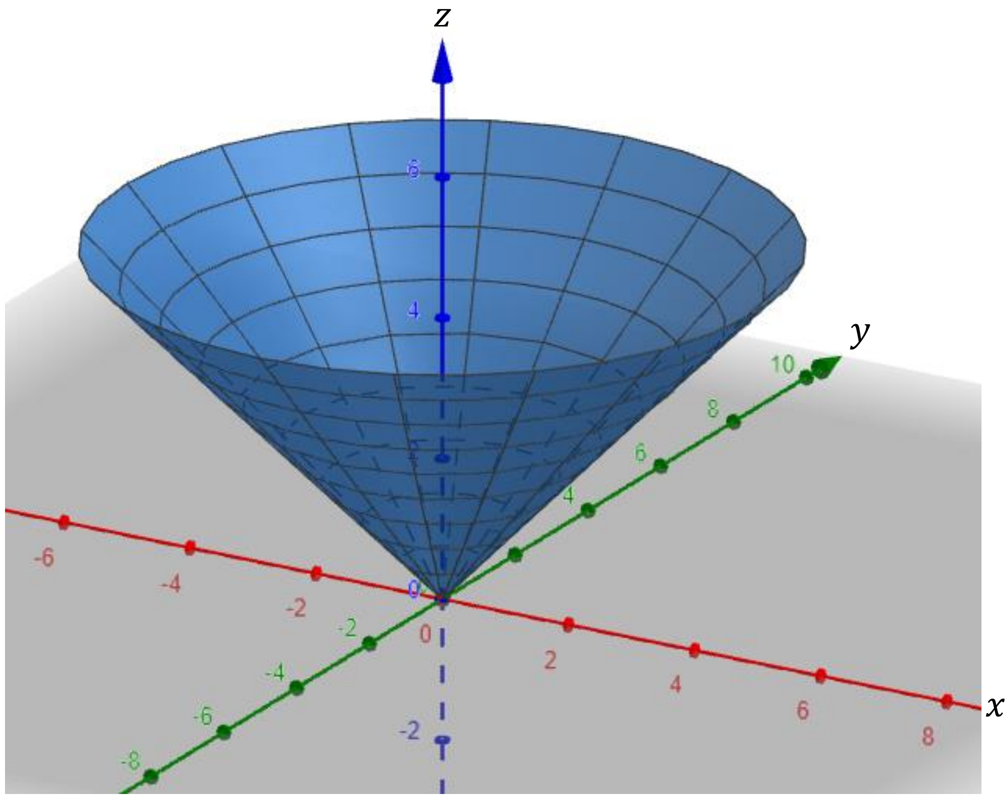
$\vec{T}_u(u_0, v_0) \times \vec{T}_v(u_0, v_0)$ is normal (perpendicular) to the surface S at $\vec{\Phi}(u_0, v_0)$.

We can use this fact to find an equation of the tangent plane to S at $\vec{\Phi}(u_0, v_0)$.

Ex. Consider the surface given by: $x = ucosv,\ y = usinv,\ z = u$; where $u \geq 0$, $0 \leq v \leq 2\pi$. Identify the surface, determine where it is smooth, and find an equation for the tangent plane at $u = 1,\ v = \frac{\pi}{2}$.

$$x^2 + y^2 = u^2(cos^2v) + u^2(sin^2v) = u^2 = z^2$$

$z^2 = x^2 + y^2$; where $u = z \geq 0$. This is the upper half of a cone about the z-axis.



$$\vec{T}_u = \frac{\partial \vec{\Phi}}{\partial u} = < \frac{\partial x}{\partial u}, \frac{\partial y}{\partial u}, \frac{\partial z}{\partial u} > = < cosv, sinv, 1 >$$

$$\vec{T}_v = \frac{\partial \vec{\Phi}}{\partial v} = < \frac{\partial x}{\partial v}, \frac{\partial y}{\partial v}, \frac{\partial z}{\partial v} > = < -usinv, ucosv, 0 >.$$

$$\vec{T}_u \times \vec{T}_v = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ cosv & sinv & 1 \\ -usinv & ucosv & 0 \end{vmatrix}$$

$$= \begin{vmatrix} sinv & 1 \\ ucosv & 0 \end{vmatrix} \vec{i} - \begin{vmatrix} cosv & 1 \\ -usinv & 0 \end{vmatrix} \vec{j} + \begin{vmatrix} cosv & sinv \\ -usinv & ucosv \end{vmatrix} \vec{k}$$

$$\vec{T}_u \times \vec{T}_v = (-ucosv)\vec{i} - (usinv)\vec{j} + u(cos^2v + sin^2v)\vec{k}$$

$$= (-ucosv)\vec{i} - (usinv)\vec{j} + u\vec{k}.$$

We want to know when $\vec{T}_u \times \vec{T}_v = 0$ (that will tell us where the surface is NOT smooth). Notice that $\vec{T}_u \times \vec{T}_v = 0$ exactly when $\left|\vec{T}_u \times \vec{T}_v\right| = 0.$

In this case that means: $\sqrt{u^2cos^2v + u^2sin^2v + u^2} = |u|\sqrt{2} = 0.$

This happens when $u = 0$. So what point(s) on the surface have $u = 0$?

$u = 0$ at the point $(0,0,0)$. So $(0,0,0)$ is the only point on S where S is NOT smooth.

To find an equation of the tangent plane at $u = 1, v = \frac{\pi}{2}$, we need to find

$\vec{T}_u \times \vec{T}_v$ at $u = 1, v = \frac{\pi}{2}$.

We know: $\vec{T}_u \times \vec{T}_v = (-ucosv)\vec{i} - (usinv)\vec{j} + u\vec{k}.$

So at $u = 1, v = \frac{\pi}{2}$, $\vec{T}_u \times \vec{T}_v = -\vec{j} + \vec{k}.$

This vector is perpendicular to the tangent plane at $u = 1, v = \frac{\pi}{2}$.

Now we need the point (x, y, z) on the surface that corresponds to $u = 1,\ v = \frac{\pi}{2}$.

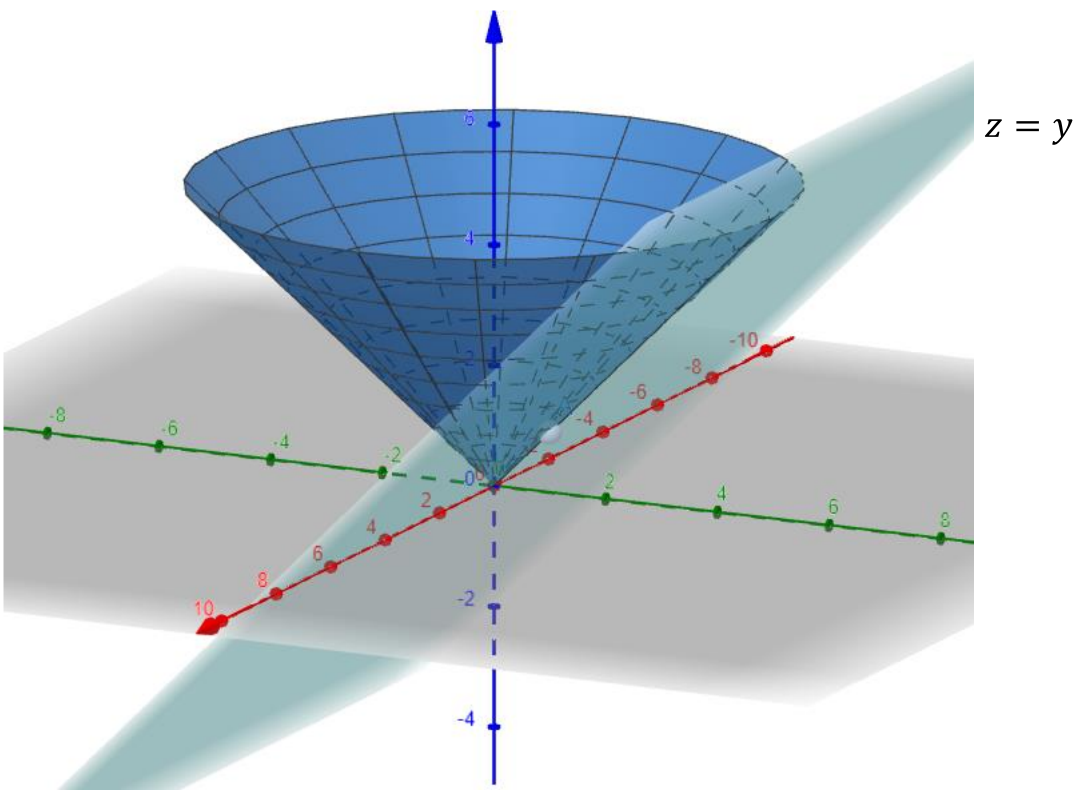
$x = ucosv, \quad y = usinv, \quad z = u\,;$ plugging in $u = 1, v = \frac{\pi}{2}$, we get:

$x = 0 \qquad y = 1 \qquad z = 1.$

Normal vector $\vec{N} = < 0, -1, 1 >$; point $= (0,1,1)$

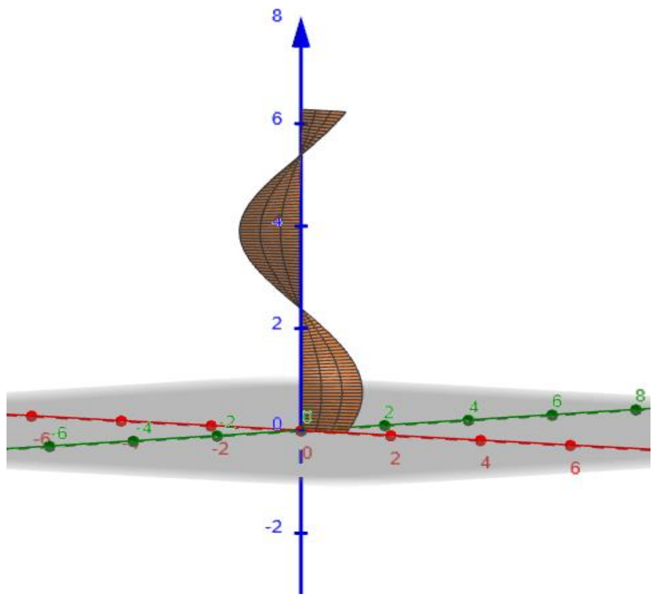
Equation of tangent plane: $0(x - 0) - 1(y - 1) + 1(z - 1) = 0$

or, $\quad -y + z = 0.$



Ex. Consider the surface in $\mathbb{R}^3$ (called a helicoid) parametrized by
$\vec{\Phi}(r,\theta) =< rcos\theta, rsin\theta, \theta >$; where $0 \le r \le 1$ and $0 \le \theta \le 2\pi$.
a) sketch the surface, b) show the surface is regular (ie smooth) everywhere,
c) find a unit normal vector at $\vec{\Phi}(r,\theta)$, and d) find an equation of the tangent plane at $r = \frac{1}{2}$ and $\theta = \frac{\pi}{4}$.

a) For any fixed value of r, $0 \le r \le 1$, $\vec{\Phi}(r,\theta) =< rcos\theta, rsin\theta, \theta >$ is just part of a helix.



b) $\vec{T}_r =< cos\theta, sin\theta, 0 >$ $\qquad$ $\vec{T}_\theta =< -rsin\theta, rcos\theta, 1 >$

$$\vec{T}_r \times \vec{T}_\theta = \begin{vmatrix} \vec{\imath} & \vec{\jmath} & \vec{k} \\ cos\theta & sin\theta & 0 \\ -rsin\theta & rcos\theta & 1 \end{vmatrix} = sin\theta\vec{\imath} - cos\theta\vec{\jmath} + r\vec{k}\ .$$

We need to show that $\vec{T}_r \times \vec{T}_\theta \neq 0$ for all $0 \le r \le 1$ and $0 \le \theta \le 2\pi$.
That's the same thing as showing that $\left|\vec{T}_r \times \vec{T}_\theta\right| \neq 0$.

$\left|\vec{T}_r \times \vec{T}_\theta\right| = \sqrt{sin^2\theta + cos^2\theta + r^2} = \sqrt{1 + r^2} \neq 0$ anywhere, so S is smooth everywhere.

c) $\vec{T}_r \times \vec{T}_\theta$ is a normal vector at $\vec{\Phi}(r, \theta)$, but its length is not 1 everywhere. So we need to divide this vector by its length to get a unit normal vector.

Unit normal$= \vec{n} = \frac{\vec{T}_r \times \vec{T}_\theta}{|\vec{T}_r \times \vec{T}_\theta|}$

$$= \frac{sin\theta\vec{\imath} - cos\theta\vec{\jmath} + r\vec{k}}{\sqrt{1+r^2}} = \frac{sin\theta}{\sqrt{1+r^2}}\vec{\imath} - \frac{cos\theta}{\sqrt{1+r^2}}\vec{\jmath} + \frac{r}{\sqrt{1+r^2}}\vec{k}.$$

d) A normal vector at $r = \frac{1}{2}$ and $\theta = \frac{\pi}{4}$ is given by evaluating $\vec{T}_r \times \vec{T}_\theta(\frac{1}{2}, \frac{\pi}{4})$.

$\vec{T}_r \times \vec{T}_\theta = sin\theta\vec{\imath} - cos\theta\vec{\jmath} + r\vec{k}$

at $r = \frac{1}{2}$ and $\theta = \frac{\pi}{4}$ we get: $\vec{T}_r \times \vec{T}_\theta = \frac{\sqrt{2}}{2}\vec{\imath} - \frac{\sqrt{2}}{2}\vec{\jmath} + \frac{1}{2}\vec{k}$.

So a normal vector to the tangent plane at $\vec{\Phi}(\frac{1}{2}, \frac{\pi}{4})$ is $< \frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}, \frac{1}{2} >$.

We need to find the point (x, y, z) that corresponds to $r = \frac{1}{2}$ and $\theta = \frac{\pi}{4}$.

$\vec{\Phi}(r, \theta) =< rcos\theta, rsin\theta, \theta >$; so $\vec{\Phi}\left(\frac{1}{2}, \frac{\pi}{4}\right) =< \frac{\sqrt{2}}{4}, \frac{\sqrt{2}}{4}, \frac{\pi}{4} >$.

Eq. of tangent plane: $\frac{\sqrt{2}}{2}\left(x - \frac{\sqrt{2}}{4}\right) - \frac{\sqrt{2}}{2}\left(x - \frac{\sqrt{2}}{4}\right) + \frac{1}{2}\left(z - \frac{\pi}{4}\right) = 0.$