Line Integrals of Vector Fields

Def. Let $\vec{F}(x, y, z)$be a vector field on $\mathbb{R}^3$ that is continuous on the C^1 curve

$c: [a, b] \to \mathbb{R}^3$. We define

$$\boldsymbol{\int_c \vec{F}(x, y, z) \cdot d\vec{s} = \int_a^b \vec{F}\left(\vec{c}(t)\right) \cdot (\overrightarrow{c'}(t))dt},$$

where $c(a)$ and $c(b)$ are the endpoints of the path c.

Notice that if $\overrightarrow{c'}(t) \neq 0$ we have:

$$\int_c \vec{F}(x, y, z) \cdot d\vec{s} = \int_a^b \vec{F}\left(\vec{c}(t)\right) \cdot (\overrightarrow{c'}(t))dt$$

$$= \int_a^b (\vec{F}\left(\vec{c}(t)\right) \cdot \frac{\overrightarrow{c'}(t)}{\left|\overrightarrow{c'}(t)\right|})(\left|\overrightarrow{c'}(t)\right|)dt$$

$$= \int_a^b \vec{F}\left(\vec{c}(t)\right) \cdot \vec{T}) (\left|\overrightarrow{c'}(t)\right|)dt; \quad \vec{T} = \text{unit tangent vector to } c.$$

So the line integral of a vector field can be thought of as the line (or path)) integral of the tangential component of $\vec{F}$ along $\vec{c}(t)$.

From elementary physics we know that if F is a constant force along the x-axis, then the work, W, done to move an object from $x = a$ to $x = b$ is $W = (F)(d)$, where $d = b - a$.

Similarly, if $\vec{F}$ is a constant force vector and $\vec{c}$ is a line segment in $\mathbb{R}^2$ or $\mathbb{R}^3$, then $W = \vec{F} \cdot \vec{d}$, where $\vec{d} = \vec{c}(t_2) - \vec{c}(t_1)$.

If $\vec{F}$ represents a force vector field then $\int_c \vec{F} \cdot d\vec{s}$ is the work done to move a particle from $c(a)$ *to* $c(b)$.

Ex. Evaluate $\int_c \vec{F}\cdot d\vec{s}$ where $\vec{F}(x,y,z) = (xy)\vec{\imath} + (yz)\vec{\jmath} + (xz)\vec{k}$, and c is given by: $c\colon [0,1] \to \mathbb{R}^3, \quad t \to < t,\ t^2,\ t^3 >$.

$\int_c \vec{F}\cdot d\vec{s}$=$\int_a^b \vec{F}\left(\vec{c}(t)\right)\cdot(\overrightarrow{c'}(t))dt$

$\vec{c}(t) =< t,\ t^2,\ t^3 >$, so $\overrightarrow{c'}(t) =< 1,\ 2t,\ 3t^2 >$, where $0 \le t \le 1$, and

$\vec{F}\left(\vec{c}(t)\right) =< t\ (t^2),\ t^2(t^3),\ t(t^3) >=< t^3,\ t^5,\ t^4 >$.

So plugging in we get:

$$\int_c \vec{F}\cdot d\vec{s} = \int_0^1 < t^3, t^5, t^4 >\cdot< 1, 2t, 3t^2 > dt$$

$$= \int_0^1 (t^3 + 2t^6 + 3t^6)dt\ = \int_0^1 (t^3 + 5t^6)dt = \tfrac{1}{4}t^4 + \tfrac{5}{7}t^7\Big|_0^1 = \frac{27}{28}.$$

Ex. Consider a force field $\vec{F} =< sinz,\ cos\sqrt{y},\ x^3 >$, find the work done to move a particle along the line segment from $(1,0,0)$ to $(0,0,3)$.

First we need to find a parametrization for the curve (a line segment) c. The line that goes through $(1,0,0)$ and $(0,0,3)$ has a direction vector:

$\vec{v} =< 0-1, 0-0, 3-0 >=< -1, 0, 3 >$;

using the point $(1,0,0)$, we get an equation for a line:

$x = 1-t,\ \ y = 0,\ \ z = 3t$; or equivalently: $\vec{c}(t) =< 1-t,\ 0,\ 3t >$.

Notice that we just want the line segment between $(1,0,0)$ and $(0,0,3)$ so we need to restrict t so that $0 \le t \le 1$.

$$\overrightarrow{c'}(t) =< -1,0,3 >; \qquad \vec{F}\big(\vec{c}(t)\big) =< sin3t, cos\sqrt{0}, (1-t)^3 >.$$

Work= $\int_c \vec{F}\cdot d\vec{s} = \int_a^b \vec{F}\left(\vec{c}(t)\right)\cdot(\overrightarrow{c'}(t))dt$

$$= \int_{t=0}^{t=1} < sin3t,\ 1,\ (1-t)^3 > \cdot < -1, 0, 3 > dt$$

$$= \int_{t=0}^{t=1} [-sin3t + 3(1-t)^3]dt$$

$$= [\tfrac{1}{3}cos3t - \tfrac{3}{4}(1-t)^4]\Big|_0^1 = \left(\tfrac{1}{3}cos3 - \tfrac{3}{4}(0)\right) - \left(\tfrac{1}{3}cos0 - \tfrac{3}{4}(1)\right)$$

$$= \frac{1}{3}cos3 - \frac{1}{3} + \frac{3}{4} = \frac{1}{3}cos3 + \frac{5}{12}.$$

Notice that:

$$d\vec{s} = \overrightarrow{c'}(t)dt$$

$$=< \frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt} > dt$$

$$=< \frac{dx}{dt}dt,\ \frac{dy}{dt}dt,\ \frac{dz}{dt}dt >$$

$$=< dx,\ dy,\ dz >.$$

Thus we can think of $d\vec{s} = (dx)\vec{\imath} + (dy)\vec{\jmath} + (dz)\vec{k}$.

And so if

$$\vec{F}(x,y,z) = F_1(x,y,z)\vec{\imath} + F_2(x,y,z)\vec{\jmath} + F_3(x,y,z)\vec{k}\,,$$

then we have:

$$\vec{F}(x,y,z) \cdot d\vec{s} = (F_1\vec{\imath} + F_2\vec{\jmath} + F_3\vec{k}) \cdot (\,(dx)\vec{\imath} + (dy)\vec{\jmath} + (dz)\vec{k})$$

$$\vec{F}(x,y,z) \cdot d\vec{s} = F_1(x,y,z)dx + F_2(x,y,z)dy + F_3(x,y,z)dz.$$

So that means we can write:

$$\boldsymbol{\int_c \vec{F} \cdot d\vec{s} = \int_c F_1(x,y,z)dx + F_2(x,y,z)dy + F_3(x,y,z)dz.}$$

In this case, represent the curve c as parametric equations,

$x = x(t),\ y = y(t),\ z = z(t)$ and calculate

$dx = x'(t)dt,\ dy = y'(t)dt,$ and $dz = z'(t)dt.$

Ex. Evaluate $\int_c y^2dx + xdy,$ where c is the line segment from $(-5-3)$ to $(0,1)$.

In this case we want to represent the curve c in parametric equations.

Let's start with the line segment c through the points $(-5,-3)$ and $(0,2)$.

Direction vector $\vec{v} =< 0-(-5), 1-(-3) >=< 5,4 >.$

Using the point $(-5,-3)$ we get:

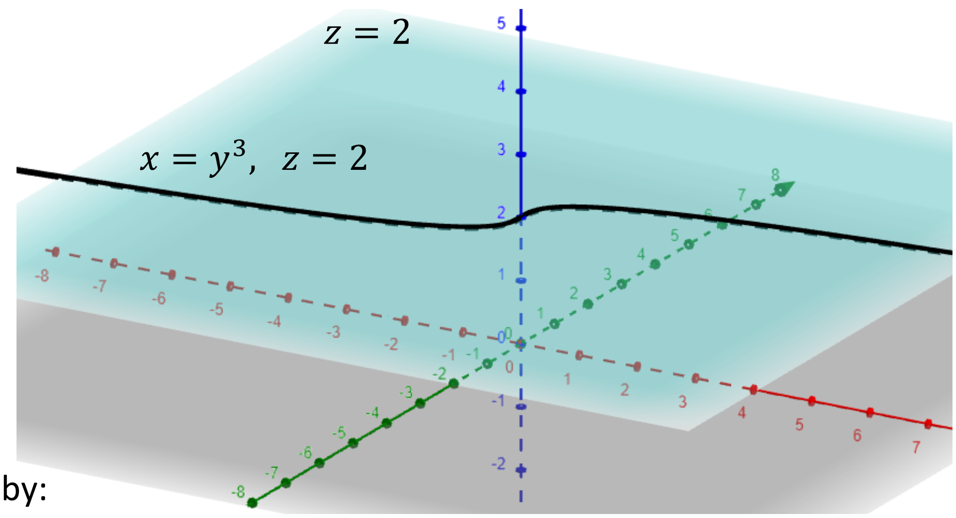
$$x = -5 + 5t,\quad y = -3 + 4t,\ \ 0 \le t \le 1,\quad dx = 5dt,\quad dy = 4dt.$$

$$\int_c y^2dx + xdy = \int_0^1[(-3+4t)^2(5)dt + (-5+5t)(4)dt]$$

$$= \int_0^1(80t^2 - 100t + 25)dt$$

$$= (\frac{80}{3}t^3 - 50t^2 + 25t)|_0^1 = \frac{80}{3} - 50 + 25 = \frac{5}{3}.$$

Ex. Find $\int_c xzdx - y^2zdy + xydz$, where c is the curve $x = y^3, z = 2$, where $-1 \le y \le 1$.



We can parametrize this curve by:

$y = t, \qquad x = y^3 = t^3, \qquad z = 2$;

$dy = dt, \quad dx = 3t^2dt, \qquad dz = 0dt.$

$$\int_c xzdx - y^2zdy + xydz =$$

$$= \int_{-1}^1(t^3)(2)(3t^2)dt - (t^2)(2)dt + (t^3)(\mathrm{t})(0)\mathrm{dt}$$

$$= \int_{-1}^1 6t^5 - 2t^2dt = -\frac{4}{3}.$$

<u>Orientation preserving vs. Orientation reversing Parametrizations</u>

Let $c_1: [a_1, b_1] \to \mathbb{R}^3$ and $c_2: [a_2, b_2] \to \mathbb{R}^3$ both be C^1, 1-1 maps with the same image in $\mathbb{R}^3$ (ie, they are the same curve). c_2 is called **Orientation Preserving** if :

$$c_2(a_2) = c_1(a_1) \text{ and } c_2(b_2) = c_1(b_1).$$

c_2 is called **Orientation Reversing** if :

$$c_2(a_2) = c_1(b_1) \text{ and } c_2(b_2) = c_1(a_1).$$

For an orientation preserving parametrization, the curve c_2 starts and ends at the same points as the curve c_1, but it may "travel at a different velocity", i.e.,

$\overrightarrow{c_1}'(t) \neq \overrightarrow{c_2}'(t)$.

Ex. Consider 3 parametrizations of the semicircle $x^2 + y^2 = 1, \; y \geq 0$:

$c_1: [0, \pi] \to \mathbb{R}^2, \;\; \overrightarrow{c_1} =< cost, sint >; \;\; 0 \leq t \leq \pi; \;\; \overrightarrow{c_1}'(t) =< -sint, cost >$

$c_2: [0, \pi] \to \mathbb{R}^2, \;\; \overrightarrow{c_2} =< -cost, sint >; \; 0 \leq t \leq \pi; \;\; \overrightarrow{c_2}'(t) =< sint, cost >$

$c_3: \left[0, \frac{\pi}{2}\right] \to \mathbb{R}^2, \;\; \overrightarrow{c_3} =< cos2t, sin2t >; \; 0 \leq t \leq \frac{\pi}{2}; \;\; \overrightarrow{c_3}'(t) =< -2sin2t, 2cos2t >$.

c_1 and c_2 have opposite orientations, i.e., c_2 is an orientation reversing parametrization of c_1.

c_1 and c_3 have the same orientation, i.e., c_3 is an orientation preserving parametrization of c_1. However, notice that the velocity vectors of c_1 and c_3 are different, ie, $\overrightarrow{c_1}'(t) \neq \overrightarrow{c_3}'(t)$.

Ex Evaluate $\int_c \vec{F}\cdot d\vec{s}$ the curves $\overrightarrow{c_1}, \overrightarrow{c_2}$, and $\overrightarrow{c_3}$ above when $\vec{F}(x,y) = (y)\vec{\imath}$.

$$\int_{c_1} \vec{F}\cdot d\vec{s} = \int_0^{\pi} < sint, 0 > \cdot < -sint, cost > dt = \int_0^{\pi} -(sin^2 t)dt$$

$$= -\int_0^{\pi}\left(\frac{1}{2} - \frac{cos2t}{2}\right)dt = -\left(\frac{1}{2}t - \frac{sin2t}{4}\right)\Big|_0^{\pi} = -\frac{\pi}{2}.$$

$$\int_{c_2} \vec{F}\cdot d\vec{s} = \int_0^{\pi} < sint, 0 > \cdot < sint, cost > dt = \int_0^{\pi}(sin^2 t)dt = \frac{\pi}{2}.$$

$$\int_{c_3} \vec{F}\cdot d\vec{s} = \int_0^{\frac{\pi}{2}} < sin2t, 0 > \cdot < -2sint, 2cost > dt$$

$$= \int_0^{\frac{\pi}{2}} -2(sin^2 2t)dt = -\frac{\pi}{2}.$$

Notice that when we changed the orientation (using c_2), the line integral became the negative of the original line integral. When we didn't change the orientation (using c_3) the value of the original integral didn't change.

Thm. Let $\vec{F}$ be a continuous vector field on the $C^1, 1-1$ curve c_1. Let c_2 be an reparametrization of the curve c_1. If c_2 is orientation preserving then:

$$\int_{c_1} \vec{F}\cdot d\vec{s} = \int_{c_2} \vec{F}\cdot d\vec{s}.$$

If c_2 is orientation reversing then:

$$\int_{c_1} \vec{F}\cdot d\vec{s} = -\int_{c_2} \vec{F}\cdot d\vec{s}.$$

The Fundamental Theorem of Calculus says: $\int_a^b f'(x)dx = f(b) - f(a)$.

There is a similar theorem for line integrals of Gradient vector fields.

Theorem: Suppose $f: \mathbb{R}^3 \to \mathbb{R}$ is a C^1, real-valued function and $c: [a,b] \to \mathbb{R}^3$ is a C^1 curve. Then

$$\int_c \overrightarrow{\nabla f} \cdot d\vec{s} = f\big(c(b)\big) - f\big(c(a)\big).$$

Notice that this says that if $\vec{F}=\overrightarrow{\nabla f}$, then $\int_c \vec{F} \cdot d\vec{s}$ depends only on the endpts of the curve c. So **any** path that is C^1 from $c(a)$ to $c(b)$ gives the same value for the integral. In this case we say the line integral is "**path independent**" (i.e., it doesn't matter how you get from $c(a)$ to $c(b)$)

Ex. Let c be the path $\vec{c}(t) =< (1-t)e^t, t^2, 0 >; \quad t \in [0,1]$. Evaluate

$\int_c 2xdx + 2ydy$. (Note: this is the same as $\int_c 2xdx + 2ydy + 0dz$).

Notice that if $\vec{F}(x,y,z) = 2x\vec{\imath} + 2y\vec{\jmath} + 0\vec{k}$; and $d\vec{s} = (dx)\vec{\imath} + (dy)\vec{\jmath} + (dz)\vec{k}$

$\int_c 2xdx + 2ydy = \int_c \vec{F} \cdot d\vec{s}$; where $\vec{F} = \nabla f$; $f(x,y,z) = x^2 + y^2$. So

$$\int_c 2xdx + 2ydy = f\big(c(1)\big) - f\big(c(0)\big)$$

$$= f(0,1,0) - f(1,0,0) = 1^2 - 1^2 = 0.$$

For now, in order to use this theorem given a vector field $\vec{F}$ we have to be able to guess a function $f(x, y, z)$ such that $\vec{F} = \nabla f$ (if such a function even exists). Later we will see a method for finding this f, if it exists.

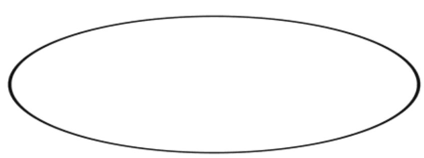
Def. A **simple curve** C is the image of a piecewise C^1 map $c: [a, b] \to \mathbb{R}^3$, that is 1-1 on [a,b]. The map c, is called a parametrization of the curve C.

Notice that a simple curve does not intersect itself (otherwise c wouldn't be 1-1).

$c(a)$ and $c(b)$ are called the endpoints of the curve C. A simple curve has 2 orientations, going from $c(a)$ to $c(b)$ and going from $c(b)$ to $c(a)$.

Def. A **simple closed curve** is a piecewise C^1 map $c: [a, b] \to \mathbb{R}^3$, that is 1-1 on $[a, b)$ and satisfies $c(a) = c(b)$.

If c is not necessarily 1-1 on $[a, b)$, and $c(a) = c(b)$, then it is called a **closed curve**.



A simple closed curve

A closed curve

A simple closed curve has 2 orientations corresponding to the 2 directions around the curve.

Once again we have the following theorem:

Thm. Let $\vec{F}$ be a continuous vector field on a simple closed curve c_1. Let c_2 be a reparametrization of the curve c_1. If c_2 is orientation preserving then:

$$\int_{c_1} \vec{F} \cdot d\vec{s} = \int_{c_2} \vec{F} \cdot d\vec{s} .$$

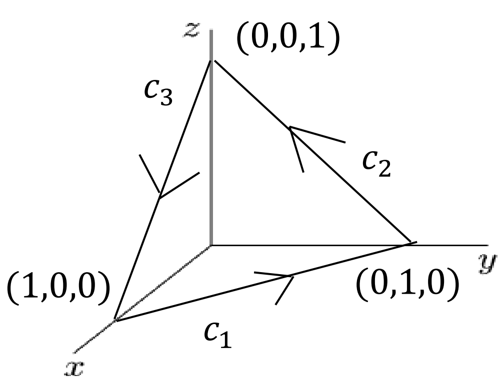
If c_2 is orientation reversing then:

$$\int_{c_1} \vec{F} \cdot d\vec{s} = -\int_{c_2} \vec{F} \cdot d\vec{s} .$$

Let c be an oriented curve consisting of several components $c_{i\,,}\ i = 1,2, \dots k,$ then we write $c = c_1 + c_2 + \cdots + c_k$ and

$$\int_{c} \vec{F} \cdot d\vec{s} = \int_{c_1} \vec{F} \cdot d\vec{s} + \int_{c_2} \vec{F} \cdot d\vec{s} + \cdots + \int_{c_k} \vec{F} \cdot d\vec{s}.$$

Ex. Find $\int_c \sin(\pi x)\, dy - \cos(\pi y) dz,$ where c is the triangle with vertices $(1,0,0), (0,1,0),$ and $(0,0,1)$ in that order.



So $c = c_1 + c_2 + c_3$, where c_1 is the line segment from $(1,0,0)$ to $(0,1,0)$, etc.

$$\int_c \sin(\pi x)\, dy - \cos(\pi y)\, dz = \int_{c_1} + \int_{c_2} + \int_{c_3} \quad .$$

We need to find parametric equations for these three line segments. First find the direction vector and then use the initial point to find the equations.

c_1: $\overrightarrow{v_1} = < -1,1,0 >$, using the point (1,0,0) we get $x = 1 - t, \ \ y = t, \ \ z = 0,$

and $dx = -dt, \quad dy = dt, \quad dz = 0.$

c_2: $\overrightarrow{v_2} = < 0,-1,1 >$, using the point (0,1,0) we get $x = 0\,, \ \ y = 1 - t, \ \ z = t\,,$

and $dx = 0, \quad dy = -dt, \quad dz = dt\,.$

c_3: $\overrightarrow{v_3} = < 1,0,-1 >$ using the point (0,0,1) we get $x = t\,, \ \ y = 0, \ \ z = \ 1 - t\,,$

and $dx = dt, \quad dy = 0\,, \quad dz = \ -dt.$

In each case $0 \le t \le 1$.

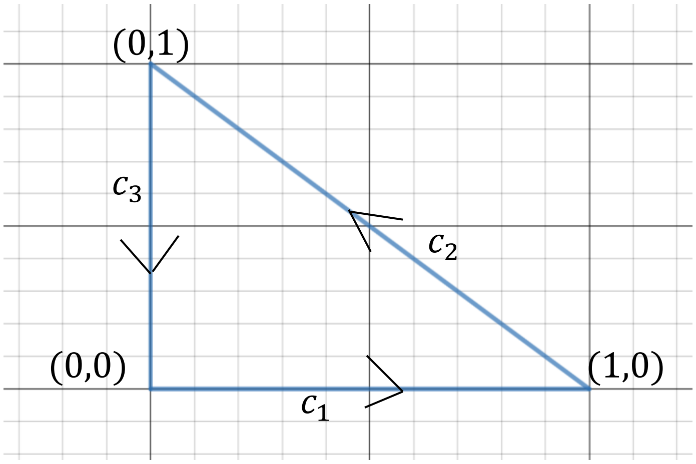
$$\int_{c_1} \sin(\pi x)\, dy - \cos(\pi y)\, dz = \int_0^1 \sin\big(\pi(1-t)\big)\,(dt) - \cos(\pi t)\,(0) = \frac{2}{\pi}$$

$$\int_{c_2} \sin(\pi x)\, dy - \cos(\pi y)\, dz = \int_0^1 \sin\big(\pi(0)\big)\,(-dt) - \cos(\pi(1-t))\,dt = 0$$

$$\int_{c_3} \sin(\pi x)\, dy - \cos(\pi y)\, dz = \int_0^1 \sin(\pi t)\,(0) - \cos(\pi(0))(-\,dt) = \ 1$$

$$\int_c \sin(\pi x)\, dy - \cos(\pi y)\, dz = \int_{c_1} + \int_{c_2} + \int_{c_3} \quad = \frac{2}{\pi} + 1.$$

Ex. Consider the triangle, c, with vertices $(0,0)$, $(1,0)$, and $(0,1)$ oriented counterclockwise. Evaluate $\int_c x^2dx + xydy$.



We need to parametrize the 3 components (sides) of the the triangle c.

$c_1: [0,1] \to \mathbb{R}^2$; So in parametric equations we have: $x = t,\ \ y = 0$;

t$\to< t, 0 >$ and therefore, $dx = dt\ \ and\ dy = 0$.

$c_2: [0,1] \to \mathbb{R}^2$; direction vector $\overrightarrow{v_2} =< -1,1 >$; $x = 1 - t,\ \ y = t$

t$\to< 1 - t, t >$ and $dx = -dt\ \ and\ \ dy = dt$.

$c_3: [0,1] \to \mathbb{R}^2$; direction vector $\overrightarrow{v_3} = < 0, -1 >$; $x = 0$, $y = 1 - t$

$t \to< 0, 1 - t >$ and $dx = 0$ $and\ \ dy = -dt$

$$\int_{c_1} x^2dx + xydy = \int_{t=0}^{t=1} t^2dt + 0 = \frac{1}{3}t^3\Big|_0^1 = \frac{1}{3}$$

$$\int_{c_2} x^2dx + xydy = \int_{t=0}^{t=1} -(1-t)^2dt + (1-t)(t)dt$$

$$= \int_{t=0}^{t=1}[-(1-2t+t^2) + t - t^2]\,dt$$

$$= \int_{t=0}^{t=1}(-1-t-2t^2)dt = -t - \frac{1}{2}t^2 - \frac{2}{3}t^3\Big|_0^1 = -\frac{13}{6}$$

$$\int_{c_3} x^2dx + xydy = \int_{t=0}^{t=1}[0^2(1) - 0(1-t)]dt = 0$$

$$\int_{c} x^2dx + xydy = \frac{1}{3} - \frac{13}{6} + 0 = -\frac{11}{6}.$$