Line Integrals

We can represent a path or curve, C, in the plane in the following 3 ways:

1. $\vec{c}(t) = < x(t), y(t) >$ (vector form of a curve)
2. $\vec{c}(t) = x(t)\vec{\imath} + y(t)\vec{\jmath}$ (another vector form of a curve)
3. $x = x(t), \quad y = y(t)$ (parametric form of a curve)

In each case t will be in some interval (eg, $a \le t \le b$)

From first year calculus we know that the length of a curve, $x = x(t), \ \ y = y(t)$, $a \le t \le b$, which we can also write as $\vec{c}(t) = < x(t), y(t) >$, is given by:

$$\text{Length of curve} = \int_c ds = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt = \int_a^b |\vec{c}\,'(t)| dt$$

where $c(a)$ and $c(b)$ are the endpoint of the curve c, $|\vec{c}\,'(t)|$ is the length of the velocity vector of $\vec{c}(t)$, and the curve is C^1, i.e., $x'(t)$, and $y'(t)$ are continuous.

We define a **line (or path) integral** of a function $f(x, y)$ over a C^1 curve c, in the plane by:

$$\int_c f(x, y) ds = \int_a^b f(x(t), y(t)) |\vec{c}\,'(t)| dt.$$

If the curve c is in 3-space rather than the plane then we have:

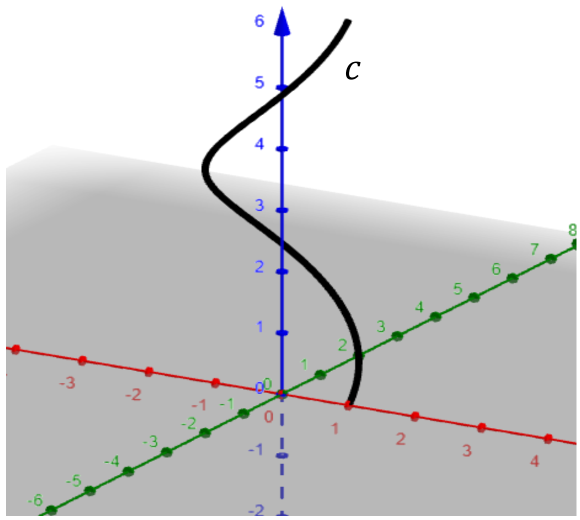
$$\int_c f(x, y, z) ds = \int_a^b f(x(t), y(t), z(t)) |\vec{c}\,'(t)| dt$$

Notice that if $f(x, y, z) = 1$, then $\int_c f(x, y, z) ds = \int_c ds = \int_a^b |\vec{c}\,'(t)| dt$

which is just the length of the curve c.

Ex. Evaluate $\int_c \; y(sinz)ds$; where c is the helix

$x(t) = cost, \;\; y(t) = sint, \;\; z(t) = t;$ where $0 \leq t \leq 2\pi.$



Notice we could write this curve as $\vec{c}(t) =< cost, \; sint, \; t >; \quad 0 \leq t \leq 2\pi.$

$\vec{c}\,'(t) =< -sint, cost, 1 > \Rightarrow \; |\vec{c}\,'(t)| = \sqrt{cos^2 t + sin^2 t + 1} = \sqrt{2}.$

$f(x(t), y(t), z(t)) = y(sinz) = (sint)(sint).$

$\int_a^b f(x(t), y(t), z(t))|\vec{c}\,'(t)|dt$

$$= \int_0^{2\pi} (sint)(sint)(\sqrt{2})dt = \sqrt{2}\int_0^{2\pi} (sin^2 t)dt.$$

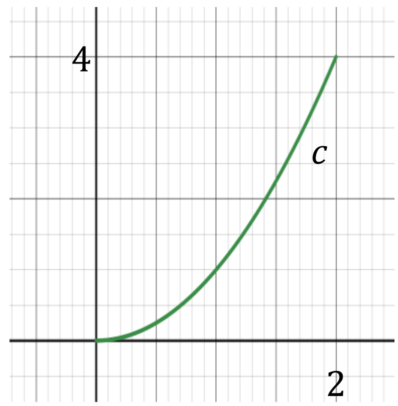
Remember: $sin^2 t = \frac{1}{2} - \frac{\cos(2t)}{2}$ and $cos^2 t = \frac{1}{2} + \frac{\cos(2t)}{2}$; we will need these formulas often!

$$\sqrt{2}\int_0^{2\pi} (sin^2 t)dt = \sqrt{2}\int_0^{2\pi} (\frac{1}{2} - \frac{\cos(2t)}{2})dt = \sqrt{2}(\frac{1}{2}t - \frac{\sin(2t)}{4}) \Big| \begin{matrix} 2\pi \\ 0 \end{matrix}$$

$$= \sqrt{2}\left[\left(\tfrac{1}{2}(2\pi) - 0\right) - (0 - 0)\right] = \sqrt{2}\pi.$$

Ex. a. Write down a definite integral that represents the length of the curve given by $c: [0,2] \to < t, t^2 >$. (or we might say $c: [0,2] \to t\vec{\imath} + (t^2)\vec{\jmath}$)

b. Find $\int_c f(x,y)ds$ where $f(x,y) = 2x$.



a. Length of a curve= $\int_c ds = \int_a^b |\vec{c}'(t)|dt$

$\vec{c}(t) = < t, t^2 >$, $\vec{c}'(t) = < 1, 2t >$, $|\vec{c}'(t)| = \sqrt{1+4t^2}$,

Length of curve= $\int_c ds = \int_a^b |\vec{c}'(t)|dt = \int_{t=0}^{t=2} \sqrt{1+4t^2}\, dt.$

b. $f\big(x(t), y(t)\big) = 2x = 2t$

$$\int_c f(x,y)ds = \int_a^b f\big(x(t), y(t)\big)|\vec{c}'(t)|dt$$

$$= \int_{t=0}^{t=2} 2t\sqrt{1+4t^2}\, dt$$

Let $u = 1 + 4t^2$ when $t = 0,\ u = 1,$

$\frac{1}{4}du = 8tdt$ when $t = 2,\ u = 17.$

$$\int_{t=0}^{t=2} 2t\sqrt{1+4t^2}\, dt = \int_{u=1}^{u=17} u^{\frac{1}{2}}\left(\frac{1}{4}\right) du = \frac{1}{4}\left(\frac{2}{3}\right) u^{\frac{3}{2}} |_1^{17}$$

$$= \frac{1}{6}\left(17^{\frac{3}{2}} - 1^{\frac{3}{2}}\right) = \frac{1}{6}\left(17\sqrt{17} - 1\right).$$

Ex. Find $\int_c xe^{yz}ds$, where c is the line segment from $(0,0,0)$ to $(1,2,3)$.

To do this problem we first need to find a parametrization of the line segment. We can find an equation of a line through the 2 points by finding the direction vector and then using either point (although (0,0,0) is easier).

Direction vector $\vec{v} =< 1-0,\ 2-0,\ 3-0 >=< 1,2,3 >$;

So an equation of the line is given by:

$x = 0 + t = t, \quad y = 0 + 2t = 2t, \quad z = 0 + 3t = 3t;$ or

$\vec{c}(t) =< t, 2t, 3t >$. We just want the line segment between $(0,0,0)$ and $(1,2,3)$. Notice that means that $0 \le t \le 1$.

$\vec{c}'(t) =< 1,2,3 >$ $\qquad |\vec{c}'(t)| = \sqrt{1^2+2^2+3^2} = \sqrt{14}$

$f\big(x(t), y(t), z(t)\big) = xe^{yz} = (t)\big(e^{(2t)(3t)}\big) = te^{6t^2}$

$$\int_c xe^{yz}ds = \int_{t=0}^{t=1} te^{6t^2}\sqrt{14}\,dt$$

Let $u = 6t^2$; $\qquad$ when $t = 0,\ u = 0$

$du = 12tdt$ $\qquad$ when $t = 1,\ u = 6$

$\frac{1}{12}du = tdt$

$$\int_c xe^{yz}ds = \int_{t=0}^{t=1} te^{6t^2}\sqrt{14}\,dt = \int_{u=0}^{u=6} e^u\left(\frac{1}{12}\right)\sqrt{14}\,du = \frac{\sqrt{14}}{12}e^u\Big|_0^6$$

$$= \frac{\sqrt{14}}{12}(e^6 - e^0) = \frac{\sqrt{14}}{12}(e^6 - 1).$$

Ex. Find $\int_c f(x,y)ds$ where $f(x,y) = xy$ and the curve is $y = x^4$ for $0 \le x \le 1$.

When we have a curve given as $y = f(x)$, we can parametrize it as

$x = t, \quad y = f(t)$.

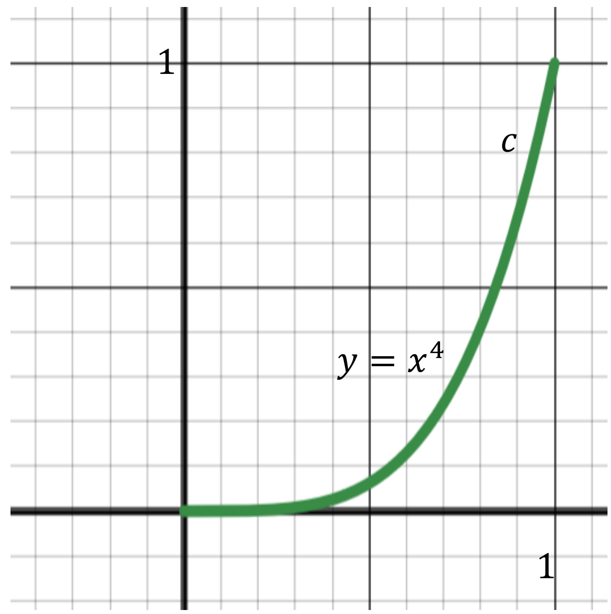
That means $\vec{c}(t) =< t, f(t) >$.

In this case: $\vec{c}(t) =< t, t^4 >$.

$\vec{c}'(t) =< 1, 4t^3 >$,

$|\vec{c}'(t)| = \sqrt{1 + 16t^6}$

$f\big(x(t), y(t)\big) = t(t^4) = t^5$



$$\int_c f(x,y)ds = \int_{t=0}^{t=1} t^5\sqrt{1 + 16t^6}dt$$

Let $u = 1 + 16t^6$ when $t = 0, \ u = 1$

$du = 96t^5 dt$ when $t = 1, \ u = 17$

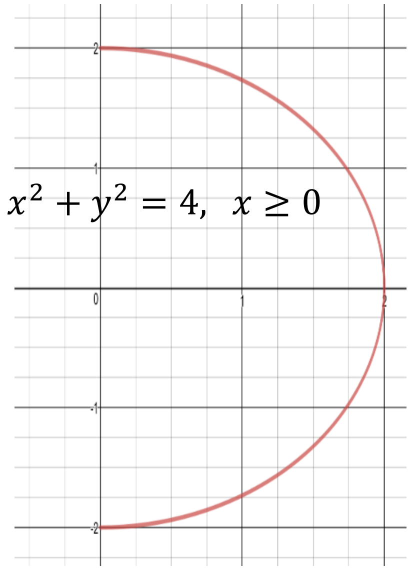
$\frac{1}{96} du = t^5 dt$

$$\int_c f(x,y)ds = \int_{t=0}^{t=1} t^5\sqrt{1 + 16t^6}dt = \int_{u=1}^{u=17} u^{\frac{1}{2}} \left(\frac{1}{96}\right) du$$

$$= \left(\frac{1}{96}\right)\left(\frac{2}{3}\right) u^{\frac{3}{2}}\Big|_1^{17} = \frac{1}{144}\left(17\sqrt{17} - 1\right).$$

Ex. Find the center of mass of a wire in the shape of a semicircle, $x^2 + y^2 = 4$,

$x \geq 0$, if the density is 5 grams/unit length.



Mass= $\int_c \rho(x,y)ds$;

where $\rho(x,y)$ is the density at a point (x,y).

The center of mass is given by $(\bar{x}, \bar{y})$, where:

$$\bar{x} = \frac{1}{mass}\int_c (x)\rho(x,y)ds,$$

$$\bar{y} = \frac{1}{mass}\int_c (y)\rho(x,y)ds.$$

Parametrize the semicircle by:

$$x = 2cost, \qquad y = 2sint, \quad -\frac{\pi}{2} \leq t \leq \frac{\pi}{2}.$$

So $\vec{c}(t) =< 2cost, 2sint >, \quad -\frac{\pi}{2} \leq t \leq \frac{\pi}{2}.$

$$\vec{c}'(t) =< -2sint, 2cost >, \quad |\vec{c}'(t)| = \sqrt{4\sin^2 t + 4\cos^2 t} = \sqrt{4} = 2$$

$$\rho(x(t), y(t)) = 5$$

Mass= $\int_C \rho(x,y)ds = \int_{t=-\frac{\pi}{2}}^{t=\frac{\pi}{2}} 5(2)dt = 10\pi.$

$$\bar{x} = \frac{1}{mass}\int_C (x)\rho(x,y)ds = \frac{1}{10\pi}\int_{t=-\frac{\pi}{2}}^{t=\frac{\pi}{2}}(2cost)5(2)dt = \frac{2}{\pi}sint|_{-\frac{\pi}{2}}^{\frac{\pi}{2}} = \frac{4}{\pi}$$

$$\bar{y} = \frac{1}{mass}\int_C (y)\rho(x,y)ds = \frac{1}{10\pi}\int_{t=-\frac{\pi}{2}}^{t=\frac{\pi}{2}}(2sint)5(2)dt$$

$$= \frac{20}{10\pi}\int_{t=-\frac{\pi}{2}}^{t=\frac{\pi}{2}}(sint)dt = -\frac{2}{\pi}cost|_{-\frac{\pi}{2}}^{\frac{\pi}{2}} = 0.$$