Gradient, Divergence, and Curl

Def. A **Vector Field** in $\mathbb{R}^n$, is a map $\vec{F}: A \subset \mathbb{R}^n \to \mathbb{R}^n$, that assigns to each point $x = (x_1, x_2, x_3, \dots, x_n) \epsilon A$, a vector $\vec{F}(x) \epsilon \mathbb{R}^n$.

If $n = 2$ we call $\vec{F}$ a vector field in the plane.

If $n = 3$ we call $\vec{F}$ a vector field in space.

We can always write a vector field in space in the form:

$\vec{F}(x, y, z) = F_1(x, y, z)\vec{\imath} + F_2(x, y, z)\vec{\jmath} + F_3(x, y, z)\vec{k}$, or

$\vec{F}(x, y, z) =< F_1(x, y, z), F_2(x, y, z), F_3(x, y, z) >$

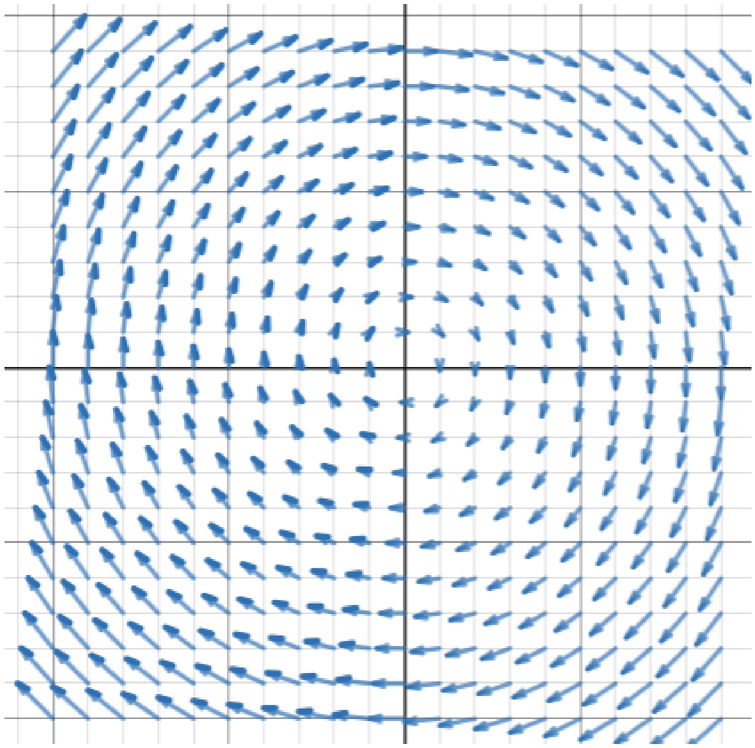
Notice that this is different from a real-valued function from $\mathbb{R}^3 \to \mathbb{R}$

(which we will sometimes call a **Scalar Field**).

Ex. $\vec{F}(x, y) = 2\vec{\imath} - \overrightarrow{3\jmath} =< 2, -3 >$ is a vector field in the plane drawn as

Ex. $\vec{F}(x,y) = y\vec{\imath} - x\vec{\jmath} = < y, -x >$ is a vector field in the plane

Point	$\vec{F}(x,y)$
$(1,0)$	$< 0, -1 >$
$(1,-1)$	$< -1, -1 >$
$(0,-2)$	$< -2, 0 >$
$(-2,-2)$	$< -2, 2 >$
$(-4,0)$	$< 0, 4 >$
$(-4,4)$	$< 4, 4 >$
$(0,8)$	$< 8, 0 >$
$(8,8)$	$< 8, -8 >$



Ex. $\vec{F}(x,y,z) = (x^2 z)\vec{\imath} + e^y\vec{\jmath} + \sin(xz)\vec{k}$ is a vector field on $\mathbb{R}^3$.

$f(x,y,z) = x^2 z + e^y + \sin(xz)$ is a real-valued function on $\mathbb{R}^3$.

Notice that for every value of x, y, z, $\vec{F}(x,y,z)$ gives us a vector in $\mathbb{R}^3$.

For every value of x, y, z, $f(x,y,z)$ gives us a real number, not a vector in $\mathbb{R}^3$.

Ex. A mass M at the origin in $\mathbb{R}^3$ exerts a force on a mass m located at

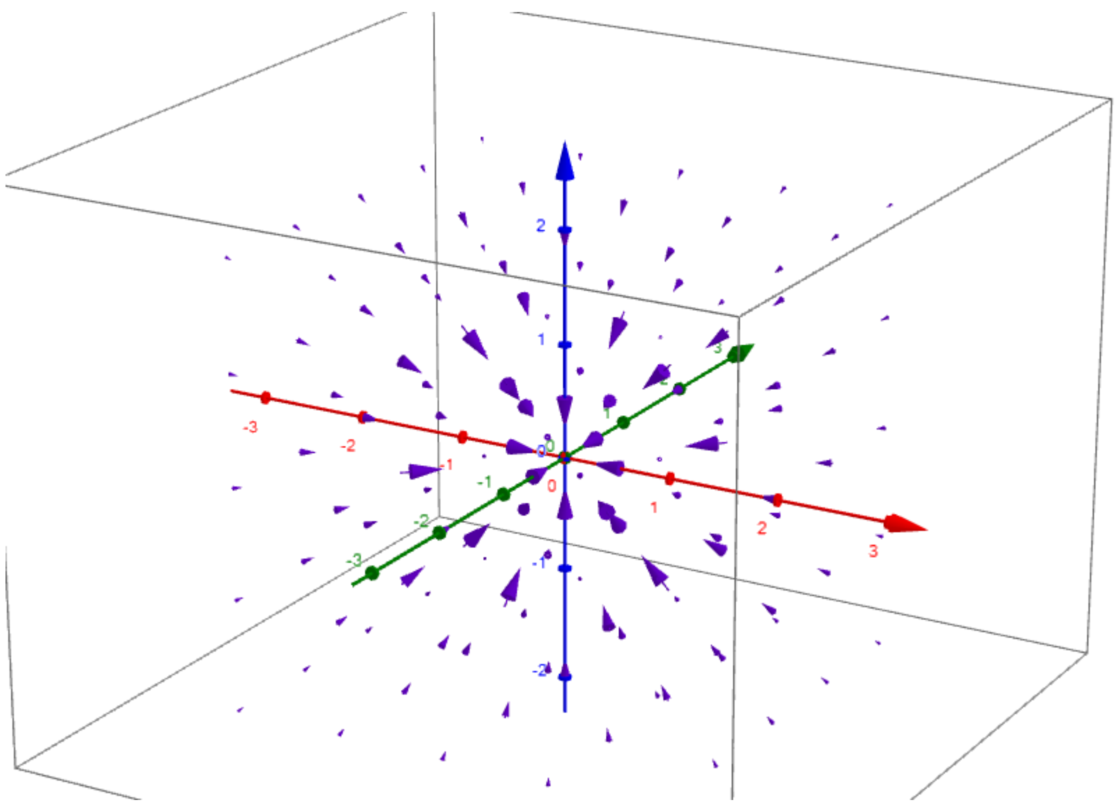
$\vec{r} =< x, y, z >$ with a magnitude of $\frac{GmM}{|\vec{r}|^2}$, where G is a gravitational constant and the direction is toward the origin. Thus we can write the force field as:

$$\vec{F}(x, y, z) = (\frac{GmM}{|\vec{r}|^2})\left(-\frac{\vec{r}}{|\vec{r}|}\right) = -(\frac{GmM}{|\vec{r}|^3})\vec{r}.$$

$$\frac{\vec{r}}{|\vec{r}|^3} =< \frac{\mathrm{x}}{(x^2+y^2+z^2)^{\frac{3}{2}}}, \frac{\mathrm{y}}{(x^2+y^2+z^2)^{\frac{3}{2}}}, \quad \frac{\mathrm{z}}{(x^2+y^2+z^2)^{\frac{3}{2}}} >$$

so we can write $\vec{F}(x, y, z)$ as:

$$\vec{F}(x, y, z) = \quad < \frac{-\mathrm{GmMx}}{(x^2+y^2+z^2)^{\frac{3}{2}}}, \frac{-\mathrm{GmMy}}{(x^2+y^2+z^2)^{\frac{3}{2}}}, \quad \frac{-\mathrm{GmMz}}{(x^2+y^2+z^2)^{\frac{3}{2}}} >$$



The **Del operator** is defined as:

$$\nabla = \vec{\imath}\frac{\partial}{\partial x} + \vec{\jmath}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}.$$

We can do 3 things with this Del operator:

1. Apply it to a real-valued function f to get the gradient(f):

$$\nabla f(x,y,z) = \frac{\partial f}{\partial x}\vec{\imath} + \frac{\partial f}{\partial y}\vec{\jmath} + \frac{\partial f}{\partial z}\vec{k} = Grad(f)$$

2. Take the dot product with a vector field $\vec{F}$ in $\mathbb{R}^3$ to get the divergence($\vec{F}$):

$$\nabla \cdot \vec{F}(x,y,z) = (\vec{\imath}\frac{\partial}{\partial x} + \vec{\jmath}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}) \cdot (F_1(x,y,z)\vec{\imath} + F_2(x,y,z)\vec{\jmath} + F_3(x,y,z)\vec{k})$$

$$\nabla \cdot \vec{F}(x,y,z) = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z} = Div(\vec{F})$$

3. Take the cross product with a vector field $\vec{F}$ in $\mathbb{R}^3$ to get the $Curl(\vec{F})$:

$$\nabla \times \vec{F}(x,y,z) = \begin{vmatrix} \vec{\imath} & \vec{\jmath} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_1 & F_2 & F_3 \end{vmatrix} = \begin{vmatrix} \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_2 & F_3 \end{vmatrix}\vec{\imath} - \begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial z} \\ F_1 & F_3 \end{vmatrix}\vec{\jmath} + \begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} \\ F_1 & F_2 \end{vmatrix}\vec{k}$$

$$\nabla \times \vec{F}(x,y,z) = \left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}\right)\vec{\imath} - \left(\frac{\partial F_3}{\partial x} - \frac{\partial F_1}{\partial z}\right)\vec{\jmath} + (\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y})\vec{k} = Curl(\vec{F}).$$

Note: we can only take the gradient of a real-valued function, **not** a vector field, and we can only take a divergence or a curl of a vector field, **not** a real-valued function.

Grad(real-valued function)= vector field

Div(vector field)=real-valued function

Curl(vector field)=vector field.

Ex. Let $\vec{F}(x,y,z) = (x^2z)\vec{\imath} + e^y\vec{\jmath} + \sin(xz)\vec{k}$ be a vector field on $\mathbb{R}^3$ and

$f(x,y,z) = x^2z + e^y + \sin(xz)$ be a real-valued function on $\mathbb{R}^3$.

Find Grad(f), Div($\vec{F}$), and Curl($\vec{F}$).

Grad(f) $= \nabla f(x,y,z) = \frac{\partial f}{\partial x}\vec{\imath} + \frac{\partial f}{\partial y}\vec{\jmath} + \frac{\partial f}{\partial z}k$

$$= (2xz + zcosxz)\vec{\imath} + e^y\vec{\jmath} + (x^2 + xcosxz)\vec{k}$$

Div($\vec{F}$)= $\nabla \cdot \vec{F}(x,y,z) = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z} = 2xz + e^y + xcosxz.$

Curl($\vec{F}$)= $\nabla \times \vec{F}(x,y,z) = \begin{vmatrix} \vec{\imath} & \vec{\jmath} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_1 & F_2 & F_3 \end{vmatrix} = \begin{vmatrix} \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_2 & F_3 \end{vmatrix}\vec{\imath} - \begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial z} \\ F_1 & F_3 \end{vmatrix}\vec{\jmath} + \begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} \\ F_1 & F_2 \end{vmatrix}\vec{k}$

$$= \left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}\right)\vec{\imath} - \left(\frac{\partial F_3}{\partial x} - \frac{\partial F_1}{\partial z}\right)\vec{\jmath} + \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right)\vec{k}$$

$$= (0 - 0)\vec{\imath} - (zcosxz - x^2)\vec{\jmath} + (0 - 0)\vec{k}$$

$$= (x^2 - zcosxz)\vec{\jmath}.$$

A vector field that is the gradient of a real-valued function is very special and is called a **Gradient Field**. One special property of gradient vector fields is:

Thm. For any C^2 function $f:\mathbb{R}^3 \to \mathbb{R}$ we have: Curl(Grad(f))=$\nabla \times \nabla(f) = 0$.

Proof:

$$\nabla \times \nabla(f) = \begin{vmatrix} \vec{\imath} & \vec{\jmath} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} & \frac{\partial f}{\partial z} \end{vmatrix} = \left(\frac{\partial^2 f}{\partial y \partial z} - \frac{\partial^2 f}{\partial z \partial y}\right)\vec{\imath} - \left(\frac{\partial^2 f}{\partial x \partial z} - \frac{\partial^2 f}{\partial z \partial x}\right)\vec{\jmath} + \left(\frac{\partial^2 f}{\partial x \partial y} - \frac{\partial^2 f}{\partial y \partial x}\right)\vec{k}$$

$$= 0.$$

Later we will see that the converse is also true: if Curl($\vec{F}$) $= 0$ then $\vec{F}$ =Grad(f) for some real-valued function f, ie, $\vec{F}$ is a gradient vector field.

Ex. Show $\vec{F}(x,y,z) = (xz)\vec{\imath} + (xyz)\vec{\jmath} - (y^2)\vec{k}$ is not a Gradient field (ie, $\vec{F} \neq \nabla f$, for some function f).

If $\vec{F}$ were a gradient vector field then by the theorem above Curl($\vec{F}$)=0. So if we can show that Curl($\vec{F}$)$\neq 0$, then $\vec{F}$ is not a gradient vector field.

$$\text{Curl}(\vec{F}) = \nabla \times \vec{F}(x,y,z) = \begin{vmatrix} \vec{\imath} & \vec{\jmath} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xz & xyz & -y^2 \end{vmatrix}$$

$$= \begin{vmatrix} \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xyz & -y^2 \end{vmatrix}\vec{\imath} - \begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial z} \\ xz & -y^2 \end{vmatrix}\vec{\jmath} + \begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} \\ xz & xyz \end{vmatrix}\vec{k}$$

$$= (-2y + xz)\vec{\imath} - (0 - x)\vec{\jmath} + (yz - 0)\vec{k} \neq \vec{0}.$$

Suppose we have a vector field in the plane, $\vec{F}(x,y) = P(x,y)\vec{\imath} + Q(x,y)\vec{\jmath}$.

We can consider it as a vector field in $\mathbb{R}^3$, where the $\vec{k}$ component is 0. In that case we could still take the Curl of $\vec{F}$ (remember, the Curl of a vector field is only defined for vector fields in $\mathbb{R}^3$, unlike the divergence which is defined on vector fields in $\mathbb{R}^n$).

$$\vec{F}(x,y) = P(x,y)\vec{\imath} + Q(x,y)\vec{\jmath} + 0\vec{k}.$$

$$\text{Curl}\big(\vec{F}\big) = \nabla \times \vec{F}(x,y,z) = \begin{vmatrix} \vec{\imath} & \vec{\jmath} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & 0 \end{vmatrix}$$

$$= \begin{vmatrix} \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ Q & 0 \end{vmatrix}\vec{\imath} - \begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial z} \\ P & 0 \end{vmatrix}\vec{\jmath} + \begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} \\ P & Q \end{vmatrix}\vec{k}$$

$$= \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\vec{k}$$

Notice that in this case, Curl($\vec{F}$) only has a $\vec{k}$ component. That component,

$\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}$, is called the **Scalar Curl of $\vec{F}$**.

Ex. Find the scalar curl of $\vec{V} = \cos(xy)\,\vec{\imath} + \sin(xy)\,\vec{\jmath}$.

$$P(x,y) = \cos(xy) \qquad\qquad Q(x,y) = \sin(xy)$$

$$\frac{\partial P}{\partial y} = -xsin(xy) \qquad\qquad \frac{\partial Q}{\partial x} = ycos(xy)$$

$$\nabla \times \vec{V}(x,y) = (ycos(xy) - \left(-xsin(xy)\right))\vec{k}$$

So the scalar Curl($\vec{V}$) $= ycos(xy) + xsin(xy)$.

Thm. For any C^2 vector field $\vec{F}$ on $\mathbb{R}^3$ we have:

Div(Curl$\vec{F}$) $= \nabla \cdot (\nabla \times \vec{F}\ )$=0.

Proof: Let $\vec{F}(x,y,z) = F_1(x,y,z)\vec{\imath} + F_2(x,y,z)\vec{\jmath} + F_3(x,y,z)\vec{k}$.

Div(Curl$\vec{F}$) $= (\vec{\imath}\frac{\partial}{\partial x} + \vec{\jmath}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z})\cdot \begin{vmatrix} \vec{\imath} & \vec{\jmath} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_1 & F_2 & F_3 \end{vmatrix}$

$$= \left(\vec{\imath}\frac{\partial}{\partial x} + \vec{\jmath}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}\right)\cdot [\left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}\right)\vec{\imath} - \left(\frac{\partial F_3}{\partial x} - \frac{\partial F_1}{\partial z}\right)\vec{\jmath} + \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right)\vec{k}]$$

$$= \left(\frac{\partial^2 F_3}{\partial x \partial y} - \frac{\partial^2 F_2}{\partial x \partial z}\right) - \left(\frac{\partial^2 F_3}{\partial y \partial x} - \frac{\partial^2 F_1}{\partial y \partial z}\right) + (\frac{\partial^2 F_2}{\partial z \partial x} - \frac{\partial^2 F_1}{\partial z \partial y})$$

$$= 0.$$

Ex. Show $\vec{V}(x,y,z) = (xz)\vec{\imath} + (xyz)\vec{\jmath} - (y^2)\vec{k}$ can't be written as the Curl of another vector field $\vec{G}$.

If $\vec{V} = Curl(\vec{G})$ then by the previous thm. Div($\vec{V}$) $= Div\left(Curl(\vec{G})\right) = 0.$

Now let's show $Div(\vec{V}) \neq 0$.

$Div(\vec{V}) = z + xz \neq \vec{0}.$

Ex. Which of the following make sense if $\vec{F}: \mathbb{R}^3 \to \mathbb{R}^3$, is a C^2 vector field and $g: \mathbb{R}^3 \to \mathbb{R}$ is a C^2, real-valued function?

a. Grad(Grad($\vec{F}$)) (no)

b. Curl(Grad(g)) (yes)

c. Grad(Div(g)) (no)

d. Grad(Div($\vec{F}$)) (yes)

e. Curl(Curl($\vec{F}$)) (yes)

f. Div(Div($\vec{F}$)) (no)

If $f: \mathbb{R}^3 \to \mathbb{R}$ is a C^2, real-valued function then

$$\nabla^2 f = \nabla \cdot \nabla(f) = (\vec{\imath}\frac{\partial}{\partial x} + \vec{\jmath}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}) \cdot (\frac{\partial f}{\partial x}\vec{\imath} + \frac{\partial f}{\partial y}\vec{\jmath} + \frac{\partial f}{\partial z}\vec{k}) =$$

$$= \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2};$$ is called the **Laplacian of f**, and is written Δf.

If $f: \mathbb{R}^2 \to \mathbb{R}$ is a C^2, real-valued function then $\Delta f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2}$.

Ex. Show $f(x,y) = e^x(siny)$ satisfies $\Delta f = 0$ (or equivalently $\nabla^2 f = 0$).

$f_x = e^x siny \qquad\qquad f_y = e^x cosy$

$f_{xx} = e^x siny \qquad\qquad f_{yy} = -e^x siny$

$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = e^x siny - e^x siny = 0.$$

Any smooth function that satifies $\Delta f = 0$ on a domain is called **harmonic** on that domain.

Important Identities of Vector Analysis:

1. $\nabla(f+g) = \nabla f + \nabla g$
2. $\nabla(cf) = c\nabla(f)$, where c is a constant
3. $Div(\vec{F}+\vec{G}) = Div(\vec{F}) + Div(\vec{G})$
4. $Curl(\vec{F}+\vec{G}) = Curl(\vec{F}) + Curl(\vec{G})$
5. $Curl(Grad(f)) = 0$
6. $Div\left(Curl(\vec{F})\right) = 0.$