Stokes' Theorem, the Divergence Theorem, and the Fundamental Theorem of Calculus

The Fundamental Theorem of Calculus is:

$$\int_a^b f'(t)dt = f(b) - f(a).$$

Let ω be a differentiable 0-form, i.e., a differentiable function $\omega = f(t)$, then

$d\omega = f'(t)dt$.

Let $I = [a, b]$ be the interval $a \leq t \leq b$. Then we define the boundary of I,

$\partial I = \{b\} - \{a\}$ (this is a formal difference of points, we don't subtract them as real number).

If we then define the zero dimensional integral of a function, f, over a point a point p to be $f(p)$ the Fundamental Theorem of Calculus becomes:

$$\int_a^b f'(t)dt = \int_I \; d\omega = \int_{\partial I} \omega = f(b) - f(a).$$

<u>Stokes' Theorem</u>

Stokes' Theorem: Let S be an oriented surface in $\mathbb{R}^3$ with a boundary consisting of a simple closed curve, ∂S, oriented as the boundary of S. Suppose that ω is a

1-form on some open set $K \subseteq \mathbb{R}^3$ that contains S then:

$$\boldsymbol{\int_{\partial S} \omega = \iint_S \; d\,\omega.}$$

Proof. Stokes' theorem said $\iint_S (\nabla \times \vec{F}) \cdot d\vec{S} = \int_{\partial S} \vec{F} \cdot d\vec{s}$.

We can write $\vec{F}$ as: $\vec{F}(x,y,z) = F_1(x,y,z)\vec{\imath} + F_2(x,y,z)\vec{\jmath} + F_3(x,y,z)\vec{k}$.

Let's let $\omega = \vec{F} \cdot d\vec{s} = F_1 dx + F_2 dy + F_3 dz$ and show that $d\omega = \left(\nabla \times \vec{F}\right) \cdot d\vec{S}$.

Then we have:

$$d\omega = d(F_1 dx + F_2 dy + F_3 dz)$$

$$= dF_1 \wedge dx + dF_2 \wedge dy + dF_3 \wedge dz$$

$$= \left(\frac{\partial F_1}{\partial x} dx + \frac{\partial F_1}{\partial y} dy + \frac{\partial F_1}{\partial z} dz\right) \wedge dx + \left(\frac{\partial F_2}{\partial x} dx + \frac{\partial F_2}{\partial y} dy + \frac{\partial F_2}{\partial z} dz\right) \wedge dy$$
$$+ \left(\frac{\partial F_3}{\partial x} dx + \frac{\partial F_3}{\partial y} dy + \frac{\partial F_3}{\partial z} dz\right) \wedge dz$$

$$= \frac{\partial F_1}{\partial y} dy \wedge dx + \frac{\partial F_1}{\partial z} dz \wedge dx + \frac{\partial F_2}{\partial x} dx \wedge dy + \frac{\partial F_2}{\partial z} dz \wedge dy$$
$$+ \frac{\partial F_3}{\partial x} dx \wedge dz + \frac{\partial F_3}{\partial y} dy \wedge dz$$

$$d\omega = \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right) dxdy + \left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}\right) dydz + \left(\frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x}\right) dzdx.$$

We need to show that $(\nabla \times \vec{F}) \cdot d\vec{S} = d\omega$.

$$\nabla \times \vec{F} = \begin{vmatrix} \vec{\imath} & \vec{\jmath} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_1 & F_2 & F_3 \end{vmatrix} = \left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}\right)\vec{\imath} + \left(\frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x}\right)\vec{\jmath} + \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right)\vec{k}.$$

To find $d\vec{S}$ we need to parametrize the surface S:

$$\vec{\Phi}(u,v) =< x(u,v), y(u,v), z(u,v) > .$$

$$d\vec{S} =(\vec{T}_u \times \vec{T}_v)dudv$$

$$\vec{T}_u = \frac{\partial\vec{\Phi}}{\partial u} = < \frac{\partial x}{\partial u},\ \frac{\partial y}{\partial u},\ \frac{\partial z}{\partial u} >, \qquad \vec{T}_v = \frac{\partial\vec{\Phi}}{\partial v} = < \frac{\partial x}{\partial v},\ \frac{\partial y}{\partial v},\ \frac{\partial z}{\partial v} >$$

$$\vec{T}_u \times \vec{T}_v = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} & \frac{\partial z}{\partial u} \\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} & \frac{\partial z}{\partial v} \end{vmatrix} = \begin{vmatrix} \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} \end{vmatrix}\vec{i} - \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} \end{vmatrix}\vec{j} + \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix}\vec{k}$$

$$= \frac{\partial(y,z)}{\partial(u,v)}\vec{i} + \frac{\partial(z,x)}{\partial(u,v)}\vec{j} + \frac{\partial(x,y)}{\partial(u,v)}\vec{k};$$

thus we have:

$$d\vec{S} =(\vec{T}_u \times \vec{T}_v)dudv = (\frac{\partial(y,z)}{\partial(u,v)}\vec{i} + \frac{\partial(z,x)}{\partial(u,v)}\vec{j} + \frac{\partial(x,y)}{\partial(u,v)}\vec{k})dudv.$$

So now we can calculate $\left(\nabla \times \vec{F}\right) \cdot d\vec{S}$:

$$\left(\nabla \times \vec{F}\right) \cdot d\vec{S} = [\left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}\right)\vec{i} + (\frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x})\vec{j} + (\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y})\vec{k}]$$

$$\cdot [\frac{\partial(y,z)}{\partial(u,v)}\vec{i} + \frac{\partial(z,x)}{\partial(u,v)}\vec{j} + \frac{\partial(x,y)}{\partial(u,v)}\vec{k}]dudv$$

$$= [\left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}\right)\frac{\partial(y,z)}{\partial(u,v)} + (\frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x})\frac{\partial(z,x)}{\partial(u,v)} + (\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y})\frac{\partial(x,y)}{\partial(u,v)}]dudv.$$

Notice that if $x = x(u,v), \quad y = y(u,v)$ then:

$$dx = \frac{\partial x}{\partial u}du + \frac{\partial x}{\partial v}dv \qquad dy = \frac{\partial y}{\partial u}du + \frac{\partial y}{\partial v}dv \qquad \text{and}$$

$$dx \wedge dy = \left(\frac{\partial x}{\partial u}du + \frac{\partial x}{\partial v}dv\right) \wedge \left(\frac{\partial y}{\partial u}du + \frac{\partial y}{\partial v}dv\right)$$

$$= \frac{\partial x}{\partial u}\frac{\partial y}{\partial v}du \wedge dv + \frac{\partial x}{\partial v}\frac{\partial y}{\partial u}dv \wedge du$$

$$= \left(\frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial x}{\partial v}\frac{\partial y}{\partial u}\right)du \wedge dv = \frac{\partial(x,y)}{\partial(u,v)}du \wedge dv.$$

Similarly, one can show:

$$dy \wedge dz = \frac{\partial(y,z)}{\partial(u,v)}du \wedge dv \quad \text{and} \quad dz \wedge dx = \frac{\partial(z,x)}{\partial(u,v)}du \wedge dv.$$

Thus our expression for $(\nabla \times \vec{F}) \cdot d\vec{S}$ becomes:

$$(\nabla \times \vec{F}) \cdot d\vec{S} = \left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}\right)dy \wedge dz + \left(\frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x}\right)dz \wedge dx + \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right)dx \wedge dy$$

$$= d\omega.$$

So we have:

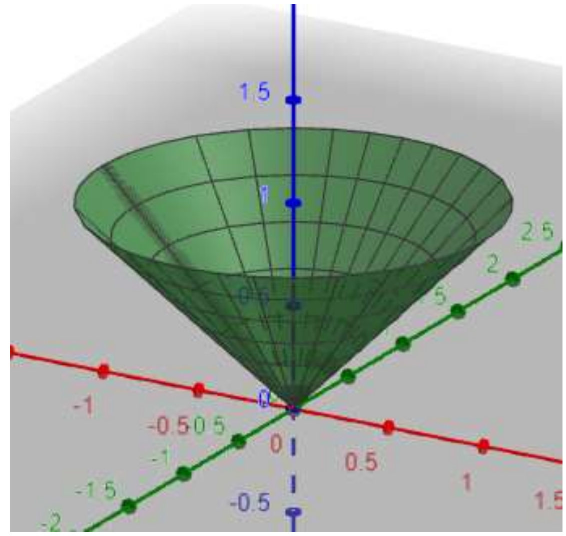
$$\iint_S (\nabla \times \vec{F}) \cdot d\vec{S} = \iint_S d\omega \quad \text{and} \quad \int_{\partial S} \vec{F} \cdot d\vec{s} = \int_{\partial S} \omega.$$

Hence we have:

$$\iint_S d\omega = \int_{\partial S} \omega.$$

Ex. Let $\omega = -yx^2dx + xy^2dy$, and let S be the portion of the cone given by

$z^2 = x^2 + y^2$, $0 \le z \le 1$. Evaluate $\int_{\partial S} \omega$ both directly and by Stokes' theorem.



Evaluating $\int_{\partial S} \omega$ directly:

∂S is the unit circle in the plane $z = 1$.

We can parametrize this by: $\vec{c}(t) = < cost, sint, 1 >$,

$0 \le t \le 2\pi$.

We then have:

$$dx = \frac{dx}{dt}dt = (-sint)dt, \qquad dy = \frac{dy}{dt}dt = (cost)dt$$

$$\int_{\partial S} \omega = \int_{\partial S} -yx^2dx + xy^2dy$$

$$= \int_0^{2\pi} -(sint)(cos^2t)(-sint)dt + (cost)(sin^2 t)(cost)dt$$

$$= \int_0^{2\pi} (sin^2t)\,(cos^2t) + (cos^2t)(sin^2t)dt = \int_0^{2\pi} 2\,(sin^2t)(cos^2t)dt$$

$$= \int_0^{2\pi} 2(\tfrac{1}{2} - \tfrac{1}{2}cos2t)\left(\tfrac{1}{2} + \tfrac{1}{2}cos2t\right)dt$$

$$= \int_0^{2\pi} (\tfrac{1}{2} - \tfrac{1}{2}cos^2 2t)dt$$

$$= \int_0^{2\pi} (\tfrac{1}{2} - \tfrac{1}{2}(\tfrac{1}{2} + \tfrac{1}{2}cos4t)dt$$

$$= \int_0^{2\pi} (\tfrac{1}{4} - \tfrac{1}{4}cos4t)dt$$

$$= (\tfrac{1}{4}t - \tfrac{1}{16}sin4t)\Big|_0^{2\pi} = \frac{\pi}{2}.$$

Now evaluate by Stokes' theorem: $\iint_S \; d\omega = \int_{\partial S} \omega$.

$$d\omega = d(-yx^2dx + xy^2dy)$$

$$= d(-yx^2) \wedge dx + (-1)^0(-yx^2) \wedge d(dx) + d(xy^2) \wedge dy$$
$$+(-1)^0(xy^2) \wedge d(dy)$$

$$= (-2xydx - x^2dy) \wedge dx + (y^2dx + 2xydy) \wedge dy$$

$$= -x^2dy \wedge dx + y^2dx \wedge dy = (x^2 + y^2)dx \wedge dy.$$

$$\iint_S \; d\omega = \iint_S \; (x^2 + y^2)dx \wedge dy.$$

To evaluate this integral we need to parametrize the cone S.

$$\vec{\Phi}(r, \theta) =< rcos\theta, rsin\theta, r >; \quad 0 \le r \le 1, \quad 0 \le \theta \le 2\pi.$$

$$dxdy = \frac{\partial(x,y)}{\partial(r,\theta)} drd\theta = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{vmatrix} drd\theta$$

$$= \begin{vmatrix} cos\theta & -rsin\theta \\ sin\theta & rcos\theta \end{vmatrix} drd\theta = rdrd\theta.$$

$$\iint_S \; d\omega = \iint_S \; (x^2 + y^2)dx \wedge dy$$

$$= \int_{\theta=0}^{2\pi} \int_{r=0}^{1} (r^2)rdrd\theta = \int_{\theta=0}^{2\pi} d\theta \int_{r=0}^{1} r^3dr = \frac{\pi}{2}.$$

<u>The Divergence Theorem</u>

Theorem (Divergence Theorem): Let $W \subset \mathbb{R}^3$ be an elementary region with ∂W given the outward orientation. If η is a 2-form on some region K which contains W, then $\iiint_W d\eta = \iint_{\partial W} \eta$.

Proof: The Divergence theorem said: $\iiint_W Div\, \vec{F} dV = \iint_{\partial W} \vec{F} \cdot d\vec{S}$.

When proving Stokes' theorem we saw that:

$$dx \wedge dy = \frac{\partial(x,y)}{\partial(u,v)} du \wedge dv$$

$$dy \wedge dz = \frac{\partial(y,z)}{\partial(u,v)} du \wedge dv$$

$$dz \wedge dx = \frac{\partial(z,x)}{\partial(u,v)} du \wedge dv$$

and that $d\vec{S} = (\vec{T}_u \times \vec{T}_v) dudv = (\frac{\partial(y,z)}{\partial(u,v)}\vec{\imath} + \frac{\partial(z,x)}{\partial(u,v)}\vec{\jmath} + \frac{\partial(x,y)}{\partial(u,v)}\vec{k})dudv$

$$= (dy \wedge dz)\vec{\imath} + (dz \wedge dx)\vec{\jmath} + (dx \wedge dy)\vec{k}\,.$$

So, $\vec{F} \cdot d\vec{S} = < F_1, F_2, F_3 > \cdot < (dy \wedge dz), (dz \wedge dx), (dx \wedge dy) >$

$$= F_1 dydz + F_2 dzdx + F_3 dxdy;$$

Now let's let η be:

$$\eta = F_1 dydz + F_2 dzdx + F_3 dxdy.$$

We need to show that $d\eta = Div\vec{F} dxdydz$.

$$d\eta = d(F_1 dydz + F_2 dzdx + F_3 dxdy)$$
$$= dF_1 \wedge dydz + (-1)^0 (F_1 \wedge d(dydz)) + dF_2 \wedge dzdx$$
$$+(-1)^0 (F_2 \wedge d(dzdx)) + dF_3 \wedge dxdy + (-1)^0 (F_3 \wedge d(dxdy)).$$

We saw earlier that $d(dydz) = d(dzdx) = d(dxdy) = 0$. So we have:

$$d\eta = dF_1 \wedge dydz + dF_2 \wedge dzdx + dF_3 \wedge dxdy$$

where:

$$dF_1 \wedge dydz = \left(\frac{\partial F_1}{\partial x} dx + \frac{\partial F_1}{\partial y} dy + \frac{\partial F_1}{\partial z} dz\right) \wedge dydz = \frac{\partial F_1}{\partial x} dxdydz$$

$$dF_2 \wedge dzdx = \left(\frac{\partial F_2}{\partial x} dx + \frac{\partial F_2}{\partial y} dy + \frac{\partial F_2}{\partial z} dz\right) \wedge dzdx = \frac{\partial F_2}{\partial y} dydzdx$$
$$= (-1)^{(2)(1)} \frac{\partial F_2}{\partial y} dxdydz = \frac{\partial F_2}{\partial y} dxdydz$$

$$dF_3 \wedge dxdy = \left(\frac{\partial F_3}{\partial x} dx + \frac{\partial F_3}{\partial y} dy + \frac{\partial F_3}{\partial z} dz\right) \wedge dxdy = \frac{\partial F_3}{\partial z} dzdxdy$$
$$= (-1)^{(2)(1)} \frac{\partial F_3}{\partial z} dxdydz = \frac{\partial F_3}{\partial z} dxdydz.$$

So now we can say:

$$d\eta = \frac{\partial F_1}{\partial x} dxdydz + \frac{\partial F_2}{\partial y} dxdydz + \frac{\partial F_3}{\partial z} dxdydz$$

$$d\eta = \left(\frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}\right) dxdydz = Div\vec{F} dxdydz.$$

Thus we have: $\iint_{\partial W} \vec{F} \cdot d\vec{S} = \iint_{\partial W} \eta$ and $\iiint_W Div\, \vec{F} dV = \iiint_W d\eta$.

So by the Divergence theorem: $\iiint_W Div\, \vec{F} dV = \iint_{\partial W} \vec{F} \cdot d\vec{S},$ we can conclude:

$$\iiint_W d\eta = \iint_{\partial W} \eta.$$

Ex. Evaluate $\iint_S \eta$, where $\eta = xdydz$ and S is the unit sphere, directly and by the Divergence Theorem.

Divergence theorem: $\iiint_W d\eta = \iint_{\partial W} \eta$ where W is the unit ball and $\partial W = S$

i.e., $W = \{(x, y, z): x^2 + y^2 + z^2 \leq 1\}$, $S = \{(x, y, z): x^2 + y^2 + z^2 = 1\}$.

$\eta = xdydz$

$d\eta = d(xdydz) = dx \wedge (dydz) + (-1)^0 x \wedge d(dydz)$

$\quad = dxdydz$

$\iiint_W d\eta = \iiint_W dxdydz = volume(W) = \frac{4}{3}\pi(1)^3 = \frac{4}{3}\pi.$

Direct calculation: $\iint_S \eta = \iint_S xdydz.$

Use the standard parametrization of the unit sphere:

$$\vec{\Phi}(u,v) = < cosvsinu, sinv\ sinu, cosu >, \quad 0 \le u \le \pi \quad , \quad 0 \le v \le 2\pi .$$

$$\iint_S xdydz = \int_{v=0}^{v=2\pi} \int_{u=0}^{u=\pi} (cosv)(sinu)(\frac{\partial(y,z)}{\partial(u,v)})dudv$$

$$\frac{\partial(y,z)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} \end{vmatrix} = \begin{vmatrix} cosu(sinv) & cosv(sinu) \\ -sinu & 0 \end{vmatrix} = sin^2u(cosv)$$

$$\iint_S xdydz = \int_{v=0}^{v=2\pi} \int_{u=0}^{u=\pi} (cosv)(sinu)(\,sin^2u)(cosv)dudv$$

$$= \int_0^{2\pi} \int_0^{\pi} cos^2 v(sin^3u)dudv = \int_0^{2\pi} (cos^2v)dv \int_0^{\pi} (sin^3u)du$$

$$= \int_0^{2\pi} (\frac{1}{2} + \frac{cos2v}{2})dv \int_0^{\pi} (sin^2u)(sinu)du$$

$$= [(\frac{1}{2}v + \frac{sin2v}{4}) \Big|_0^{2\pi}] \int_0^{\pi} (1 - cos^2u)(sinu)du$$

$$= (\pi)(-cosu + \frac{cos^3u}{3} \Big|_0^{\pi} = (\pi)(\frac{4}{3}) = \frac{4}{3}\pi.$$

General Stokes' Theorem: Let M be a (compact) oriented n-dimensional manifold with boundary and ω is a $(n-1)$-form on M then:

$$\int_M d\omega = \int_{\partial M} \omega.$$