Differential Forms

Differential forms will allow us to express the fundamental theorem of calculus, Green's theorem, Stokes' theorem, and the divergence theorem all as the same theorem.

We have already encountered differential 1-forms (which were called differentials in the first 3 semesters of calculus). For example, if $z = f(x, y)$ we have:

$$dz = \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy.$$

So if $z = x^2 + xsiny$,

$$dz = (2x + siny)dx + (xcosy)dy.$$

The expression $(2x + siny)dx + (xcosy)dy$ is called a **differential 1-form** (or just a 1-form for short).

Real valued functions on an open set in $\mathbb{R}^3$ (or $\mathbb{R}^n$) are called **0-forms**. Thus when we take the differential of a function (a 0-form) we get a 1-form. In fact, we will see that we can define the operation of taking a differential of an n-form to get an $(n + 1)$-form. We will then see that the fundamental theorem of calculus, Green's theorem, Stokes' theorem, and the divergence theorem can all be written as:

$$\int_{\partial M} \omega = \int_M d\omega,$$

where ω is a differential n-form and $d\omega$ (the differential of ω) is an $n + 1$ form.

For the purposes of this section we will assume that all functions have as many derivatives as we need.

<u>0-Forms</u>

Let K be an open set in $\mathbb{R}^3$. A zero form on K is a real valued function $f: K \to \mathbb{R}$. Given two 0-forms f_1 and f_2 on K, we can add them or multiply them.

Ex. Let $f_1(x, y, z) = xe^{yz} + 2xy$, $f_2(x, y, z) = xy$. Then we have:

$$f_1(x, y, z) + f_2(x, y, z) = xe^{yz} + 2xy + xy = xe^{yz} + 3xy$$

$$[f_1(x, y, z)][f_2(x, y, z)] = (xe^{yz} + 2xy)(xy) = x^2ye^{yz} + 2x^2y^2$$

<u>1-Forms</u>

A 1-form on $K \subseteq \mathbb{R}^3$ is of the form :

$\omega = P(x, y, z)dx + Q(x, y, z)dy + R(x, y, z)dz,$ or

$\omega = Pdx + Qdy + Rdz.$

We line integrate 1-forms over a curve.

If we write: $\omega = Q(x, y, z)dy,$ this is just a 1-form where $P(x, y, z) = R(x, y, z) = 0.$

It also doesn't matter which order we write the terms in:

$$\omega = Pdx + Qdy + Rdz = Qdy + Rdz + Pdx.$$

However, the standard form is:

$$\omega = Pdx + Qdy + Rdz.$$

Given two 1-forms we can add them:

$$\omega = Pdx + Qdy + Rdz$$

$$\eta = Ldx + Mdy + Ndz$$

$$\omega + \eta = (P + L)dx + (Q + M)dy + (R + N)dz$$

Ex. $\omega = (x^2y + 3y)dx + xzdy - xydz$

$$\eta = (x - 2y)dx + (x^2 - xz)dy + 4xydz$$

$$\omega + \eta = (x^2y + x + y)dx + x^2dy + 3xydz.$$

If $f(x, y, z)$ is a 0-form (ie a function), we can multiply a 1-form by a 0-form and get a new 1-form.

Ex. $f(x, y, z) = x^4$

$\omega = xyzdx + e^{yz}dy + x^3dz;$ then we have:

$$f\omega = x^5yzdx + x^4e^{yz}dy + x^7dz.$$

There will be a way to multiply two 1-forms, but we will see that later.

2-Forms

A 2-form on $K \subseteq \mathbb{R}^3$ is of the form:

$$\omega = F(x, y, z)dxdy + G(x, y, z)dydz + H(x, y, z)dzdx;$$

where $F, G, H: K \to \mathbb{R}$.

The order in which we add the terms does not matter:

$$Fdxdy + Gdydz + Hdzdx = Gdydz + Hdzdx + Fdxdy.$$

However, the order of dx, dy, dz **does matter**. We will come back to this.

For now we will write all 2-forms in the standard form:

$$\omega = F(x,y,z)dxdy + G(x,y,z)dydz + H(x,y,z)dzdx\ .$$

To add two 2-forms, we add their corresponding components.

$$\omega = Fdxdy + Gdydz + Hdzdx$$

$\eta = Ldxdy + Mdydz + Ndzdx$; then we have:

$$\omega + \eta = (F+L)dxdy + (G+M)dydz + (H+N)dzdx$$

As with 1-forms, we can multiply a 2-form by a 0-form and get a new 2-form.

Ex. $\omega = (y^2 - x)dxdy + xdydz - x^3z^2dzdx$

$$\eta = 3xdxdy + ydydz + yzdzdx$$

$f(x,y,z) = xyz^2$; then we have:

$$\omega + \eta = (y^2 + 2x)dxdy + (x+y)dydz + (yz - x^3z^2)dzdx$$

$$f\omega = (xy^3z^2 - x^2yz^2)dxdy + x^2yz^2dydz - x^4yz^4dzdx.$$

There is a way to multiply 2-forms and 1-forms, but we will see that later.

3-Forms

A 3-form on $K \subseteq \mathbb{R}^3$ in standard form:

$\omega = f(x, y, z)dxdydz$;   where $f: K \subseteq \mathbb{R}^3 \to \mathbb{R}$.

We can add 3-forms and we can multiply 3-forms by a 0-form.

Ex.
$$\omega = (x + y^2 + z^3)dxdydz$$
$$\eta = (x - 2y^2 + z^3)dxdydz$$

$f(x, y, z) = yz^2$;   then we have:

$$\omega + \eta = (2x - y^2 + 2z^3)dxdydz$$
$$f\omega = (xyz^2 + y^3z^2 + yz^5)dxdydz.$$

Again the order of dx, dy, dz matters. $dxdydz \neq dydxdz$.

Rules of Multiplication of Differential Forms

Let ω be a k-form and η be an l-form , $0 \leq k + l \leq 3$, on $K \subseteq \mathbb{R}^3$, there is a product called the **wedge product**, $\omega \wedge \eta$, that is a $k + l$ form on $K \subseteq \mathbb{R}^3$. This wedge product satisfies the following properties:

1. For each k, there is a zero k-form, 0, such that:

   $0 + \omega = \omega$, for all k-forms ω, and $0 \wedge \eta = 0$, l-forms η if $0 \leq k + l \leq 3$.

2. (Distributive property) If f is a zero-form then:

   $(f\omega_1 + \omega_2) \wedge \eta = f(\omega_1 \wedge \eta) + (\omega_2 \wedge \eta)$.

3. (Anti-commutativity) $\omega \wedge \eta = (-1)^{kl}(\eta \wedge \omega)$

   Ex. When we write $\omega = x^2 dxdy$, we mean $\omega = x^2 dx \wedge dy$.

$$(dx) \wedge (dy) = (-1)^{(1)(1)} dy \wedge dx = -dy \wedge dx$$

$$(dx) \wedge (dydz) = (-1)^{(1)(2)}(dydz) \wedge (dx) = dydzdx.$$

$$(dx) \wedge (dx) = (-1)^{(1)(1)} dx \wedge dx = -dx \wedge dx$$

which means $dx \wedge dx = 0$.

This also means: $dy \wedge dy = 0$ and $dz \wedge dz = 0$.

4. (Associativity) If $\omega_1, \omega_2, \omega_3$ are k_1, k_2, k_3 forms respectively with $k_1 + k_2 + k_3 \leq 3$ then

$$\omega_1 \wedge (\omega_2 \wedge \omega_3) = (\omega_1 \wedge \omega_2) \wedge \omega_3.$$

5. If f is a zero-form, then

   $\omega \wedge (f\,\eta) = (f\omega) \wedge \eta$ and $f \wedge \omega = f\omega$.

Ex. Let $\omega = zdx + y^2dz$ and $\eta = (x)dxdy$. Find $\omega \wedge \eta$ and $\eta \wedge \omega$.

$$\omega \wedge \eta = (zdx + y^2dz) \wedge (xdxdy)$$

$$= (zdx) \wedge (xdxdy) + (y^2dz \wedge xdxdy) \qquad \text{(distributive property)}$$

$$= zx(dx \wedge dxdy) + xy^2(dz \wedge dxdy) \qquad \text{(property 5)}$$

$$= zx(dx \wedge dx \wedge dy) + xy^2(dz \wedge dxdy) \qquad (dxdy = dx \wedge dy)$$

$$= zx(0) + xy^2((-1)^{(1)(2)}(dxdy) \wedge (dz) \qquad \text{(Anti-commutativity)}$$

$$= xy^2dxdydz$$

$$\eta \wedge \omega = (-1)^{(1)(2)}(\omega \wedge \eta) = \omega \wedge \eta = xy^2dxdydz.$$

So in this case: $\omega \wedge \eta = \eta \wedge \omega.$

Ex. Let $\omega = ydx - xdy$ and $\eta = xydx + y^2zdy + ydz$. Find $\omega \wedge \eta$.

$$\omega \wedge \eta = (ydx - xdy) \wedge (xydx + y^2zdy + ydz) \qquad \text{(now distribute)}$$

$$= xy^2dx \wedge dx + y^3zdx \wedge dy + y^2dx \wedge dz$$

$$-x^2ydy \wedge dx - xy^2zdy \wedge dy - xydy \wedge dz.$$

Now write each term in terms of either $dx \wedge dy,\ dy \wedge dz,\ or\ dz \wedge dx$.

$$= xy^2(0) + y^3zdx \wedge dy - y^2dz \wedge dx + x^2ydx \wedge dy - xy^2(0) - xydy \wedge dz$$

$$\omega \wedge \eta = (y^3z + x^2y)dx \wedge dy - xydy \wedge dz - y^2dz \wedge dx.$$

Ex. Let $\omega = (x^2y)dx + (xz)dz$ and $\eta = (x)dxdy - (yz)dydz$. Find $\omega \wedge \eta$.

$\omega \wedge \eta = (x^2ydx + xzdz) \wedge (xdxdy - yzdydz)$ (now distribute)

$= x^3ydx \wedge dxdy - x^2y^2zdx \wedge dydz + x^2zdz \wedge dxdy - xyz^2dz \wedge dydz$

Notice that $dx \wedge dxdy = dx \wedge dx \wedge dy = 0$, because $dx \wedge dx = 0$ and

$dz \wedge dydz = dz \wedge dy \wedge dz = (-1)^{(1)(1)}dy \wedge dz \wedge dz = 0$, since $dz \wedge dz = 0$

$\omega \wedge \eta = -(x^2y^2z)dxdydz + x^2z(-1)^{1(2)}dxdy \wedge dz$

$= (x^2z - x^2y^2z)dxdydz.$

The Differential, d, of a Differential Form ω

We now need to define $d\omega$, the differential of a differential form ω.

1. If $f: K \to \mathbb{R}$ is a 0-form (i.e., a real valued function), then

$$df = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy + \frac{\partial f}{\partial z}dz$$

2. If ω_1 and ω_2 are k-forms, then

$$d(\omega_1 + \omega_2) = d\omega_1 + d\omega_2$$

3. If ω is a k-form and η is an l-form then

$$d(\omega \wedge \eta) = (d\omega \wedge \eta) + (-1)^k(\omega \wedge d\eta)$$

Note that the exponent depends only on what form ω is and not η.

4. $d(d\omega) = 0$ (so in particular, $d(dx) = 0,\ \ d(dy) = 0,\ \ d(dz) = 0$).

The differntial of a k-form is a $(k+1)$ −form.

Ex. Let $\omega = x^2ydx + z^2dy$, find $d\omega$.

$$d\omega = d(x^2ydx + z^2dy) = d(x^2ydx) + d(z^2dy)$$ (by #2, now apply #3)

$$= d(x^2y) \wedge dx + (-1)^0(x^2y \wedge d(dx)) + d(z^2) \wedge dy + (-1)^0(z^2 \wedge d(dy))$$

$$= \left(\frac{\partial(x^2y)}{\partial x}dx + \frac{\partial(x^2y)}{\partial y}dy + \frac{\partial(x^2y)}{\partial z}dz\right) \wedge dx - (x^2y \wedge 0)$$

$$+ \left(\frac{\partial(z^2)}{\partial x}dx + \frac{\partial(z^2)}{\partial y}dy + \frac{\partial(z^2)}{\partial z}dz\right) \wedge dy - z^2 \wedge 0$$ (by #1 and #4)

$$= (2xydx + x^2dy + 0dz) \wedge dx + (0dx + 0dy + 2zdz) \wedge dy$$

$$= 2xydx \wedge dx + x^2dy \wedge dx + 2zdz \wedge dy$$ (distributive property)

$$= 0 - x^2dx \wedge dy - 2z\, dy \wedge dz = -x^2dx \wedge dy - 2z\, dy \wedge dz\ .$$

Ex. Let $\eta = xz^2 dxdy + ye^x dydz,$ find $d\eta$.

$$d\eta = d(xz^2 dxdy + ye^x dydz)$$

$$= d(xz^2 dxdy) + d(ye^x dydz) \qquad \text{(by \#2)}$$

$$= d(xz^2) \wedge (dxdy) + (-1)^0\big(xz^2 \wedge d(dxdy)\big)$$
$$+d(ye^x) \wedge (dydz) + (-1)^0\big(ye^x \wedge d(dydz)\big) \qquad \text{(by \#3)}$$

$$=\left(\frac{\partial(xz^2)}{\partial x}dx + \frac{\partial(xz^2)}{\partial y}dy + \frac{\partial(xz^2)}{\partial z}dz\right) \wedge (dxdy) + (xz^2 \wedge d(dxdy))$$
$$+\left(\frac{\partial(ye^x)}{\partial x}dx + \frac{\partial(ye^x)}{\partial y}dy + \frac{\partial(ye^x)}{\partial z}dz\right) \wedge (dydz) + (ye^x \wedge d(dydz)) \qquad \text{(by \#1)}$$

Now notice: $d(dx \wedge dy) = d(dx) \wedge dy + (-1)^1 dx \wedge d(dy) = 0$ (by #3,#4)

$$d(dy \wedge dz) = d(dy) \wedge dz + (-1)^1 dy \wedge d(dz) = 0.$$

$$d\eta = (\, z^2 dx + 0dy + 2xzdz) \wedge (dxdy) + (ye^x dx + e^x dy + 0dz) \wedge (dydz)$$

$$= z^2 dx \wedge dxdy + (2xz)dz \wedge dxdy + \; ye^x dx \wedge dydz + e^x dy \wedge dydz$$

(distr. prop)

$$= 0 + (2xz)dzdxdy + (ye^x)dxdydz + 0$$

$$= (2xz(-1)^{(1)(2)})dxdydz + (ye^x)dxdydz = (2xz + ye^x)dxdydz.$$

Ex. Let $\omega = P(x,y)dx + Q(x,y)dy$, be a 1-form on some open set $K \subseteq \mathbb{R}^2$.
Find $d\omega$.

$$d\omega = d(P(x,y)dx + Q(x,y)dy)$$

$$= d(P(x,y)dx) + d(Q(x,y)dy) \qquad \text{(by \#2)}$$

$$= dP \wedge dx + (-1)^0 P \wedge d(dx) + dQ \wedge dy + (-1)^0 Q \wedge d(dy) \qquad \text{(by \#3)}$$

$$= \left(\frac{\partial P}{\partial x} dx + \frac{\partial P}{\partial y} dy\right) \wedge dx + 0 + \left(\frac{\partial Q}{\partial x} dx + \frac{\partial Q}{\partial y} dy\right) \wedge dy$$

$$= \frac{\partial P}{\partial x} dx \wedge dx + \frac{\partial P}{\partial y} dy \wedge dx + \frac{\partial Q}{\partial x} dx \wedge dy + \frac{\partial Q}{\partial y} dy \wedge dy \qquad (\#1, \#4)$$

$$= 0 - \frac{\partial P}{\partial y}\, dx \wedge dy + \frac{\partial Q}{\partial x} dx \wedge dy + 0 \qquad \text{(anti-comm)}$$

$$= \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx \wedge dy.$$

This should look familiar from Green's theorem:

$$\int_{\partial D} P(x,y)dx + Q(x,y)dy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)dxdy.$$

In other words, if $\omega = P(x,y)dx + Q(x,y)dy$ and thus
$d\omega = \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx \wedge dy$, then Green's theorem can be written:

$$\boldsymbol{\int_{\partial D} \omega = \iint_D d\,\omega.}$$

Ex. Let $\omega = f(x,y,z)dxdydz$ on an open set $K \subseteq \mathbb{R}^3$. Show $d\omega = 0$.

$$d\omega = d(f(x,y,z)dxdydz)$$

$$= df \wedge dxdydz + (-1)^0 f(x,y,z) \wedge d(dx \wedge dydz) \qquad \text{(by \#3)}$$

$$= \left(\frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy + \frac{\partial f}{\partial z}dz\right) \wedge dxdydz$$

$$+ f \wedge [d(dx) \wedge dydz + (-1)^1 d(dydz)]. \qquad \text{(by \#1, \#3)}$$

$$= 0.$$

Since $\left(\frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy + \frac{\partial f}{\partial z}dz\right) \wedge dxdydz = 0$

because each term will be a 4-form where we will have either $dx \wedge dx, dy \wedge dy,$ $or\ \ dz \wedge dz,$ in it, all of which are 0 and

The second term is 0 because: $d(dx) = 0$ by #4, and $d(dydz) = 0$ (see the 2nd example, page 10 in this section). Thus, $d\omega = 0$.