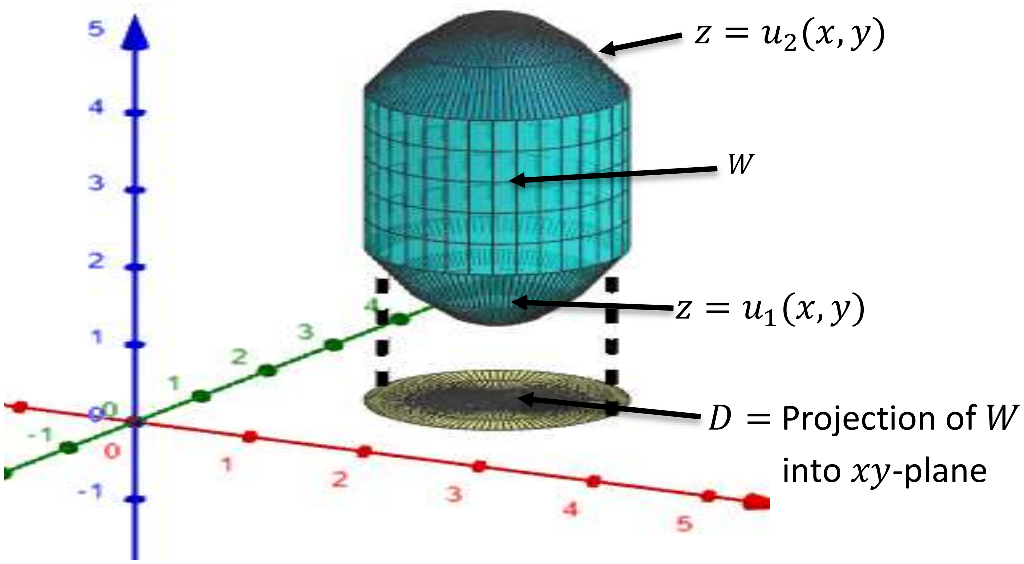
The Divergence Theorem (Gauss' Theorem)

Gauss's divergence theorem says that the flux of a vector field out of a closed surface equals the integral of the divergence over the volume enclosed by the surface, ie, $\iint_{\partial W} \vec{F} \cdot d\vec{S} = \iiint_W Div\, \vec{F} dV$; where ∂W is a closed surface.

We begin by looking at Elementary Regions (eg, W below), ie regions bounded by, for example, $z = u_1(x, y)$ and $z = u_2(x, y)$, whose projection, D, into the xy-plane is bounded by $y = g_1(x)$ and $y = g_2(x)$, $x = a$ and $x = b$.



Ex. A cube is an elementary region (as is a sphere). Let's take the cube W

$0 \le x \le 1,\ \ 0 \le y \le 1,\ \ 0 \le z \le 1.$

S_1: $z = 0,\ \ 0 \le x \le 1,\ \ 0 \le y \le 1$

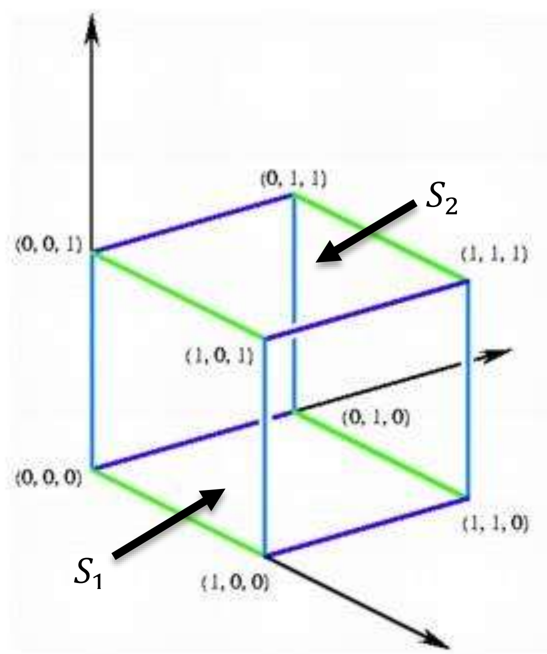
S_2: $z = 1,\ \ 0 \le x \le 1,\ \ 0 \le y \le 1$

S_3: $x = 0,\ \ 0 \le y \le 1,\ \ 0 \le z \le 1$

S_4: $x = 1,\ \ 0 \le y \le 1,\ \ 0 \le z \le 1$

S_5: $y = 0,\ \ 0 \le x \le 1,\ \ 0 \le z \le 1$

S_6: $y = 1,\ \ 0 \le x \le 1,\ \ 0 \le z \le 1$



$\iint_S \vec{F} \cdot d\vec{S} = \sum_{i=1}^{6} \iint_{S_i} \vec{F} \cdot d\vec{S}_i = \sum_{i=1}^{6} \iint_{S_i} (\vec{F} \cdot \vec{n}_i) dS_i$

Since $d\vec{S}_i = (\vec{n}_i) dS_i$; $\quad dS_i = |\vec{T}_u \times \vec{T}_v| du dv$

In the case of this unit cube, $\vec{n}_1 = -\vec{k}$, $\vec{n}_2 = \vec{k}$, $\vec{n}_3 = -\vec{\imath}$, $\vec{n}_4 = \vec{\imath}$, etc.

So, for example, $\iint_{S_3} (\vec{F} \cdot \vec{n}_3) dS_3 = \iint_{S_i} (\vec{F}) \cdot (-\vec{\imath}) dS_3 = \iint_{S_i} -F_1 dS_3$.

Theorem (Gauss' Divergence Theorem) Let W be a symmetric elementary region in $\mathbb{R}^3$. Denote ∂W as the oriented closed surface that bounds W. Let $\vec{F}$ be a smooth vector field on W. Then

$$\iiint_W Div\,\vec{F} dV = \iiint_W \nabla \cdot \vec{F} dV = \iint_{\partial W} \vec{F} \cdot d\vec{S} = \iint_{\partial W} (\vec{F} \cdot \vec{n}) dS.$$

Note: The Divergence theorem can be proved for more general regions that can be broken up into a finite number of symmetric elementary regions.

Outline of proof:

If $\vec{F} = P(x,y,z)\vec{\imath} + Q(x,y,z)\vec{\jmath} + R(x,y,z)\vec{k}$; then

$Div\vec{F} = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}$; so

$\iiint_W Div\,\vec{F} dV = \iiint_W \frac{\partial P}{\partial x} dV + \iiint_W \frac{\partial Q}{\partial y} dV + \iiint_W \frac{\partial R}{\partial z} dV$.

However,

$$\iint_{\partial W} (\vec{F} \cdot \vec{n}) dS = \iint_{\partial W} (P\vec{\imath} + Q\vec{\jmath} + R\vec{k}) \cdot \vec{n} dS$$

$$= \iint_{\partial W} (P\vec{\imath}) \cdot \vec{n} dS + \iint_{\partial W} (Q\vec{\jmath}) \cdot \vec{n} dS + \iint_{\partial W} (R\vec{k}) \cdot \vec{n} dS.$$

We can show $\iiint_W Div\,\vec{F}dV = \iint_{\partial W}(\vec{F}\cdot\vec{n})dS$ by showing

$$\iiint_W \frac{\partial P}{\partial x}dV = \iint_{\partial W}(P\vec{\imath})\cdot\vec{n}dS$$

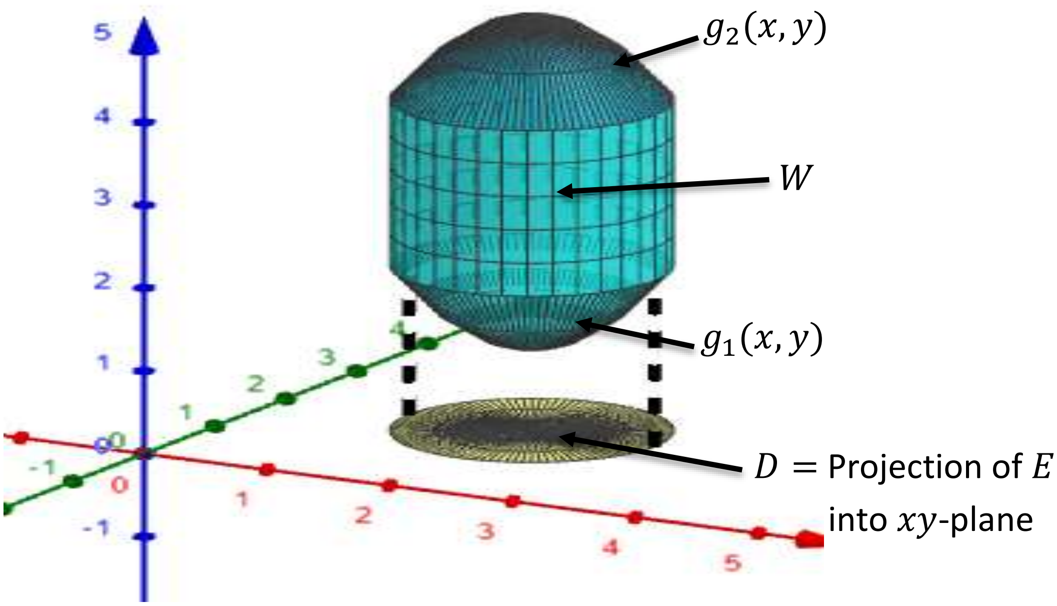
$$\iiint_W \frac{\partial Q}{\partial y}dV = \iint_{\partial W}(Q\vec{\jmath})\cdot\vec{n}dS$$

$$\iiint_W \frac{\partial R}{\partial z}dV = \iint_{\partial W}(R\vec{k})\cdot\vec{n}dS\ .$$

Let's start with $\iiint_W \frac{\partial R}{\partial z}dV = \iint_{\partial W}(R\vec{k})\cdot\vec{n}dS$.

Since W is a symmetric elementary region there are functions $z = g_1(x,y)$,

$z = g_2(x,y)$ and an elementary region D in the xy-plane such that W is given by $g_1(x,y) \le z \le g_2(x,y);\ \ (x,y) \in D$.



Thus we see that:

$$\iiint_W \frac{\partial R}{\partial z}dV = \iint_D \int_{z=g_1(x,y)}^{z=g_2(x,y)} \left(\frac{\partial R}{\partial z}\right)dzdxdy$$

$$= \iint_D \left(R\big(x,y,g_2(x,y)\big) - R(x,y,g_1(x,y)\right)dxdy.$$

To show that we get the same thing for $\iint_{\partial W}(R\vec{k})\cdot\vec{n}dS$, notice that we can think of ∂W as made up of 6 surfaces, $S_1, S_2, \dots, S_6$, where S_1 is the "bottom" boundary of W, S_2 is the "top", and $S_3, \dots, S_6$ are the "sides". But for each of the "sides" $\vec{n}$ is perpendicular to $\vec{k}$, so $\vec{n}\cdot\vec{k} = 0$. Thus we get:

$$\iint_{\partial W}(R\vec{k})\cdot\vec{n}dS = \iint_{S_1}(R\vec{k})\cdot\vec{n}dS + \iint_{S_2}(R\vec{k})\cdot\vec{n}dS.$$

Remembering that the normal for ∂W points outward (so the normal for S_1 points downward), we can calculate:

$d\overrightarrow{S_1} = \vec{n}dS_1 = \frac{\partial g_1}{\partial x}\vec{\imath} + \frac{\partial g_2}{\partial y}\vec{\jmath} - \vec{k}$. Thus we get:

$$\iint_{S_1}(R\vec{k})\cdot\vec{n}dS = -\iint_D R\big(x,y,g_1(x,y)\big)dxdy.$$

Similarly, we get:

$$\iint_{S_2}(R\vec{k})\cdot\vec{n}dS = \iint_D R\big(x,y,g_2(x,y)\big)dxdy.$$

$$\iint_{\partial W}(R\vec{k})\cdot\vec{n}dS = \iint_D R\big(x,y,g_2(x,y)\big)dxdy - \iint_D R\big(x,y,g_1(x,y)\big)dxdy$$

$$= \iiint_W \frac{\partial R}{\partial z}dV.$$

$\iiint_W \frac{\partial P}{\partial x}dV = \iint_{\partial W}(P\vec{\imath})\cdot\vec{n}dS$ and $\iiint_W \frac{\partial Q}{\partial y}dV = \iint_{\partial W}(Q\vec{\jmath})\cdot\vec{n}dS$ are done analogously.

Ex. Find the flux of the vector field $\vec{F}(x,y,z) = (ye^{z})\vec{\imath} + (xy)\vec{\jmath} - (xz)\vec{k}$ over the unit sphere $x^2 + y^2 + z^2 = 1$, i.e., find $\iint_S \vec{F} \cdot d\vec{S}$.

Notice that $Div\vec{F} = 0 + x - x = 0$. So by the divergence theorem we have:

$\iint_S \vec{F} \cdot d\vec{S} = \iiint_W Div\,\vec{F} dV = \iiint_W 0\, dV = 0$, where W is the unit ball $x^2 + y^2 + z^2 \leq 1$.

Ex. Consider the vector field $\vec{F}(x,y,z) = (z)\vec{\imath} + (y)\vec{\jmath} + (z)\vec{k}$. Let S be the unit sphere $x^2 + y^2 + z^2 = 1$. Evaluate $\iint_S (\vec{F} \cdot \vec{n}) dS$.

By the divergence theorem: $\iint_S (\vec{F} \cdot \vec{n}) dS = \iiint_W \nabla \cdot \vec{F} dV$, where W is the unit ball $x^2 + y^2 + z^2 \leq 1$.

Notice that $\nabla \cdot \vec{F} = Div\vec{F} = 0 + 1 + 1 = 2$. So we have:

$$\iint_S (\vec{F} \cdot \vec{n}) dS = \iiint_W \nabla \cdot \vec{F} dV = \iiint_W 2\, dV$$

$$= 2\big(volume(W)\big) = 2\left(\tfrac{4}{3}\pi(1)^3\right) = \frac{8\pi}{3} .$$

Ex. Verify the Divergence theorem for $W = \{(x, y, z): x^2 + y^2 + z^2 \leq 1\}$, and

$\vec{F}(x, y, z) = (x)\vec{\imath} + (y)\vec{\jmath} + (z)\vec{k}$.

We need to show that $\iint_{\partial W} \vec{F} \cdot d\vec{S} = \iiint_W Div\, \vec{F} dV$.

Let's start with the RHS. $Div\vec{F} = 1 + 1 + 1 = 3$. So we have:

$$\iiint_W Div\, \vec{F} dV = \iiint_W 3\, dV = 3(vol\ of\ W) = 3(\frac{4}{3}\pi(1)^3) = 4\pi.$$

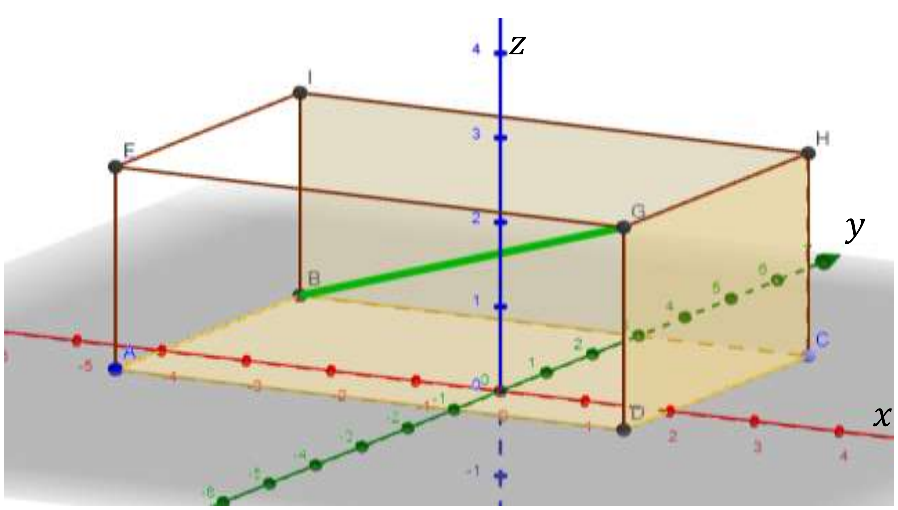
LHS: ∂W is the unit sphere. To calculate $\iint_{\partial W} \vec{F} \cdot d\vec{S}$, we could parametrize the unit sphere and calculate $\vec{T}_\phi \times \vec{T}_\theta$, but it's much easier in this case to use:

$\iint_{\partial W} \vec{F} \cdot d\vec{S} = \iint_{\partial W} (\vec{F} \cdot \vec{n}) dS$, where $\vec{n} = \frac{(x)\vec{i}+(y)\vec{j}+(z)\vec{k}}{\sqrt{x^2+y^2+z^2}} = (x)\vec{\imath} + (y)\vec{\jmath} + (z)\vec{k}$.

$$\iint_{\partial W} (\vec{F} \cdot \vec{n}) dS = \iint_{\partial W} < x, y, z > \cdot < x, y, z > dS = \iint_{\partial W} (x^2 + y^2 + z^2) dS$$

$= \iint_{\partial W} (1) dS$ =Surface area of the unit sphere=4π.

Ex. Find the flux of the vector field $\vec{F}(x, y, z) =< z - y^2, x, xyz >$ out of the rectangular solid $[-4,2]x[-1,3]x[0,3]$ (this is W).



The flux is given by $\iint_{\partial W} \vec{F} \cdot d\vec{S}$. By the divergence theorem we know:

$$\iint_{\partial W} \vec{F} \cdot d\vec{S} = \iiint_{W} Div\, \vec{F} dV.$$

$$Div\vec{F} = 0 + 0 + xy = xy.$$

$$\iint_{\partial W} \vec{F} \cdot d\vec{S} = \iiint_{W} xy\, dV = \int_{x=-4}^{x=2} \int_{y=-1}^{y=3} \int_{z=0}^{z=3} (xy)\, dzdydx$$

$$= \int_{x=0}^{x=1} \int_{y=-1}^{y=2} xyz \ \Big|_0^3 dydx$$

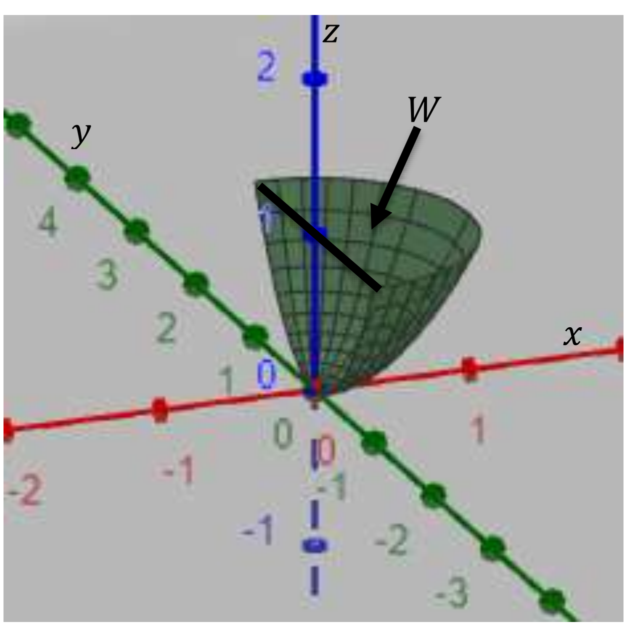
$$= \int_{x=-4}^{x=2} \int_{y=-1}^{y=3} 3xydydx = \int_{x=-4}^{2} \frac{3}{2} xy^2 \Big|_{-1}^{3} dx$$

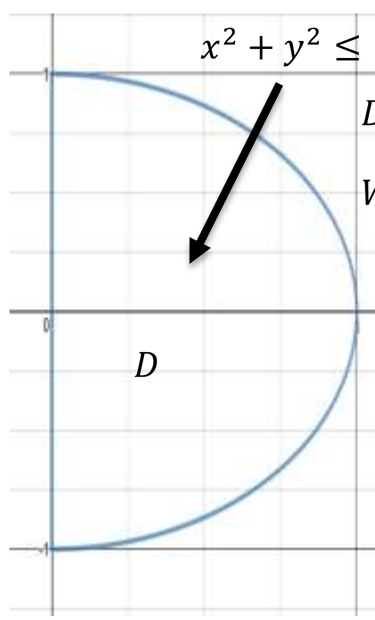
$$= \int_{x=-4}^{x=2} 12xdx = -72.$$

Ex. $\vec{F}(x, y, z) = (y)\vec{\imath} + (z)\vec{\jmath} + (xz)\vec{k}$, Evaluate $\iint_{\partial W} \vec{F} \cdot d\vec{S}$, if W is given by

$x^2 + y^2 \leq z \leq 1, \quad x \geq 0.$

First draw the region W given by $x^2 + y^2 \leq z \leq 1, \ \ x \geq 0.$





By the divergence theorem we know:

$$\iint_{\partial W} \vec{F} \cdot d\vec{S} = \iiint\limits_{W} Div\, \vec{F} dV.$$

$$Div\vec{F} = 0 + 0 + x = x.$$

$\iint_{\partial W} \vec{F} \cdot d\vec{S} = \iiint_{W} x\, dV.$ Now we have to integrate over the region W.

$$\iint_{\partial W} \vec{F} \cdot d\vec{S} = \iiint\limits_{W} x\, dV = \iint_{\substack{x^2+y^2 \le 1 \\ x \ge 0}} \int_{z=x^2+y^2}^{1} xdzdA$$

$$\iint_{\partial W} \vec{F} \cdot d\vec{S} = \iint_{\substack{x^2+y^2 \le 1 \\ x \ge 0}} xz \Big|_{x^2+y^2}^{1} dA$$

$$= \iint_{\substack{x^2+y^2 \le 1 \\ x \ge 0}} x(1 - x^2 - y^2) dA.$$

Now change to polar coordinates: $x = rcos\theta, \quad y = rsin\theta, \quad dA = rdrd\theta.$

$$= \int_{\theta=-\frac{\pi}{2}}^{\theta=\frac{\pi}{2}} \int_{r=0}^{r=1} (rcos\theta)(1 - r^2) rdrd\theta$$

$$= \left(\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} cos\theta\, d\theta\right) \int_{r=0}^{r=1} (r^2 - r^4) drd\theta$$

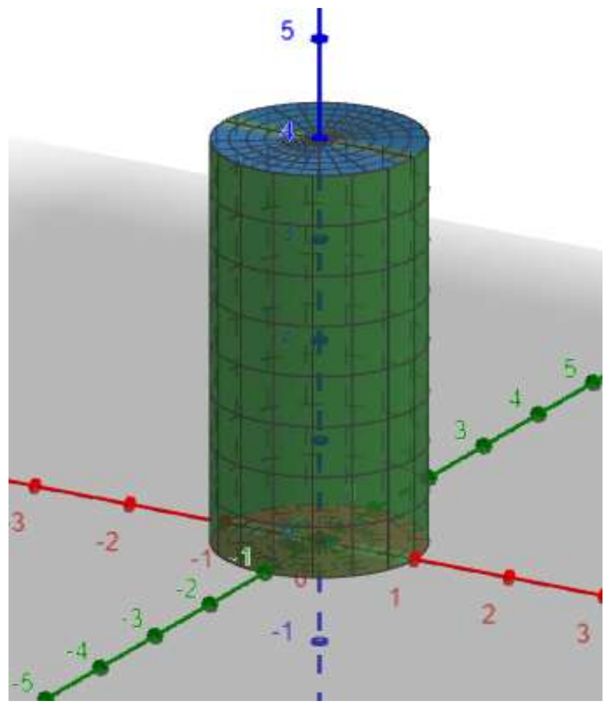
$$= \left[sin\theta \Big|_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\right] \left[\left(\frac{r^3}{3} - \frac{r^5}{5}\right) \Big|_0^1\right]$$

$$= (2)\left(\frac{1}{3} - \frac{1}{5}\right) = \frac{4}{15}.$$

Ex. Evaluate $\iint_{\partial S}(\vec{F}\cdot\vec{n})dA$, where $\vec{F}(x,y,z)=(z)\vec{\imath}+(x-z)\vec{\jmath}+(z(x^2+y^2))\vec{k}$

And ∂S is the surface which is the boundary of the portion of the solid cylinder given by $x^2+y^2\leq 4,\;\; 0\leq z\leq 4$.

First draw the solid S.



By the divergence theorem:

$$\iint_{\partial S}(\vec{F}\cdot\vec{n})dA=\iiint_S\; Div\,\vec{F}dS.$$

$Div\vec{F}=x^2+y^2$, so we have:

$$\iint_{\partial S}(\vec{F}\cdot\vec{n})dA=\iiint_S\; Div\,\vec{F}dS$$

$$=\iiint_S\;(x^2+y^2)\,dS.$$

We are integrating over the region S which is part of a solid cylinder, thus we change to cylindrical coordinates to do the integral (which is just polar coordinates plus a z coordinate).

$$x=r cos\theta,\quad y=r sin\theta,\quad z=z;\qquad dS=rdzdrd\theta.$$

$$\iiint_S(x^2+y^2)\,dS=\int_{\theta=0}^{\theta=2\pi}\int_{r=0}^{r=2}\int_{z=0}^{z=4}(r^2)(r)dzdrd\theta$$

$$=\int_{\theta=0}^{\theta=2\pi}\int_{r=0}^{r=2}\int_{z=0}^{z=4}(r^3)dzdrd\theta$$

$$=\int_{\theta=0}^{2\pi}\int_{r=0}^{2}r^3z\;\Big|_{z=0}^{z=4}drd\theta$$

$$= \int_{\theta=0}^{2\pi} \int_{r=0}^{2} 4r^3 \, dr d\theta$$

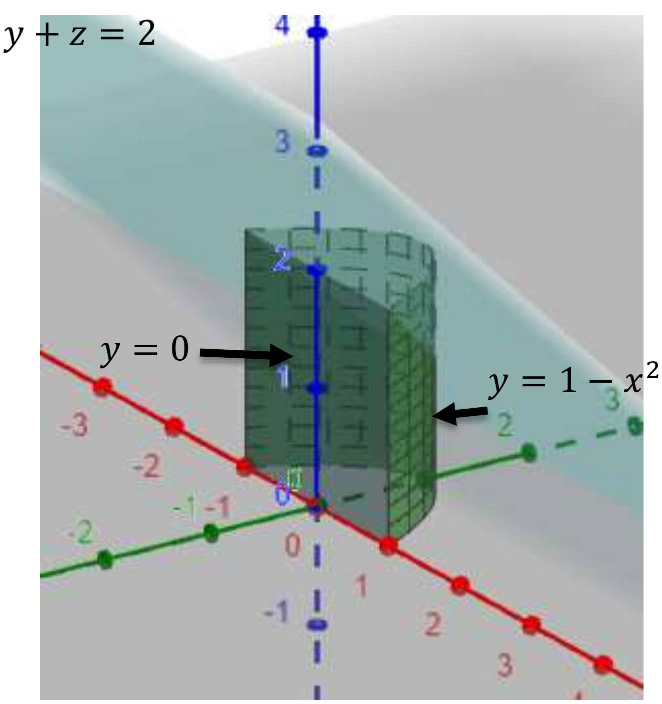
$$= \int_{\theta=0}^{2\pi} r^4 \Big|_{r=0}^{r=2} d\theta$$

$$= \int_{\theta=0}^{2\pi} 16 d\theta = 32\pi.$$

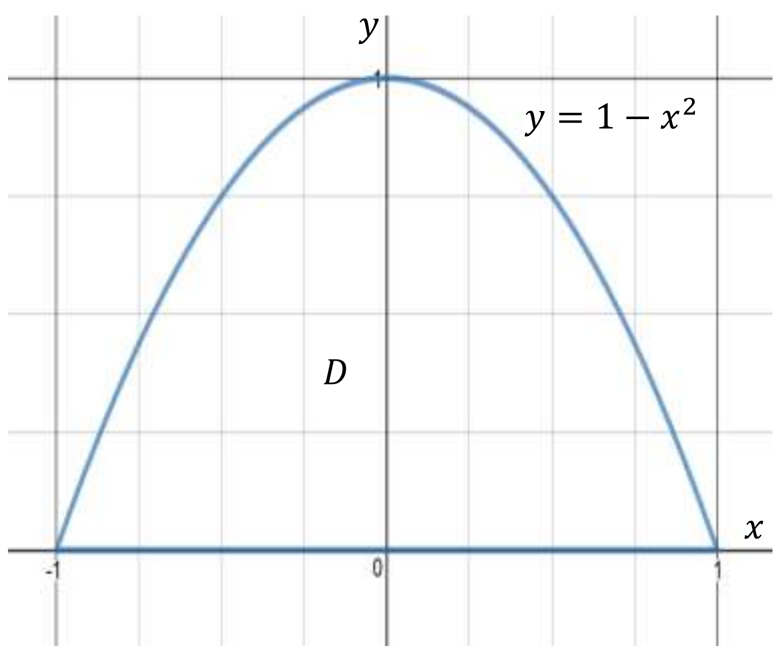
Ex. Evaluate $\iint_S \vec{F} \cdot d\vec{S}$ where $\vec{F}(x, y, z) = < xz, \sin(z), z^2 + e^{xy^2} >$ and S is the surface that's the boundary of the solid W, where S is made up of the surfaces:

The parabolic cylinder $y = 1 - x^2$, and the planes $z = 0$, $y = 0$, and $y + z = 2$.

First draw the surface S.



D, Projection of W into xy-plane



$$\iint_S \vec{F} \cdot d\vec{S} = \iiint_W Div\, \vec{F} dV = \iiint_W 3z\, dV$$

$$= \int_{x=-1}^{x=1} \int_{y=0}^{y=1-x^2} \int_{z=0}^{z=2-y} 3z dz dy dx$$

$$= \int_{x=-1}^{x=1} \int_{y=0}^{y=1-x^2} \frac{3}{2} z^2 \Big|_{z=0}^{z=2-y} dy dx$$

$$= \int_{x=-1}^{x=1} \int_{y=0}^{y=1-x^2} \frac{3}{2} (2-y)^2 dy dx$$

$$= \int_{x=-1}^{x=1} -\frac{1}{2} (2-y)^3 \Big|_{y=0}^{y=1-x^2} dx$$

$$= \int_{x=-1}^{x=1} -\frac{1}{2} \left[\left(2-(1-x^2)\right)^3 - (2-0)^3 \right] dx$$

$$= -\frac{1}{2} \int_{x=-1}^{x=1} ((x^2+1)^3 - 8) dx$$

$$= -\frac{1}{2} \int_{x=-1}^{x=1} (x^6 + 3x^4 + 3x^2 - 7) dx = \frac{184}{35}.$$

Ex. Find the flux of the vector field

$\vec{F}(x,y,z) = <\frac{1}{3}x^3 + e^{\sqrt{y}}, \sin(x^2) + z, y^2 + xy>$ out of the unit sphere,

$x^2 + y^2 + z^2 = 1$.

$\iint_{\partial W} \vec{F} \cdot d\vec{S} = \iiint_W Div\,\vec{F} dV$ where W is given by $x^2 + y^2 + z^2 \le 1$.

$Div\vec{F} = x^2$.

So we must evaluate $\iiint_W x^2\, dV$ over the unit ball $x^2 + y^2 + z^2 \le 1$.

Method #1: change to spherical coordinates.

$x = \rho cos\theta sin\phi, \quad y = \rho sin\theta sin\phi, \quad z = \rho cos\phi,$

where $0 \le \rho \le 1,\ 0 \le \theta \le 2\pi,\ 0 \le \phi \le \pi$.

From 3rd semester Calculus we know that $dV = (\rho^2 sin\phi) d\rho d\phi d\theta$.

$$\iiint_W x^2\, dV = \int_{\theta=0}^{\theta=2\pi} \int_{\phi=0}^{\phi=\pi} \int_{\rho=0}^{\rho=1} (\rho^2 \cos^2\theta \sin^2\phi)(\rho^2 sin\phi)\, d\rho d\phi d\theta.$$

$$= \int_{\theta=0}^{\theta=2\pi} \int_{\phi=0}^{\phi=\pi} \int_{\rho=0}^{\rho=1} (\rho^4 \cos^2\theta \sin^3\phi)\, d\rho d\phi d\theta$$

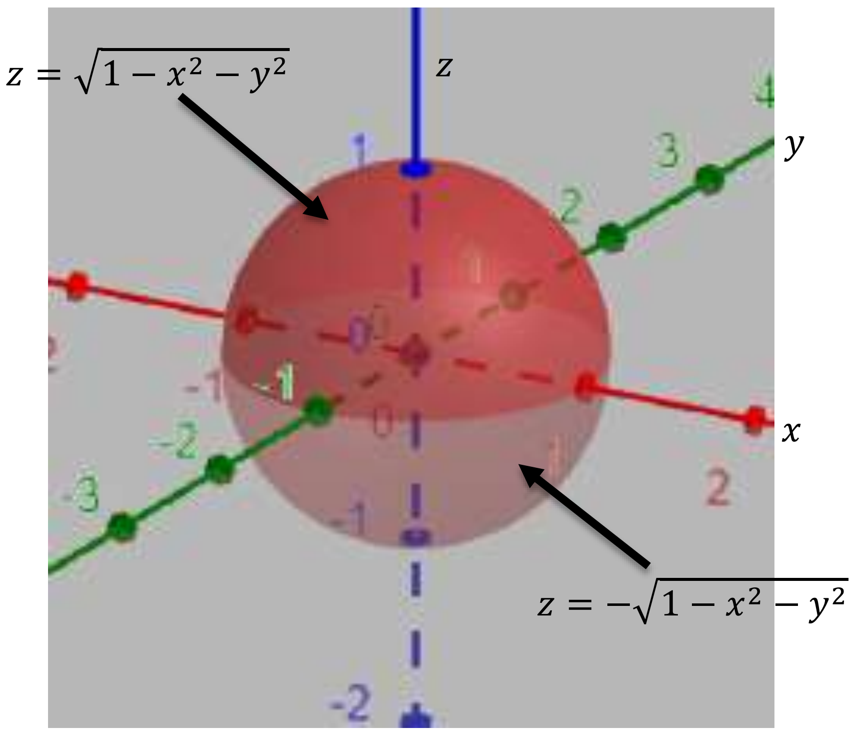
$$= \left(\int_{\theta=0}^{2\pi} \cos^2\theta d\theta\right) \left(\int_{\phi=0}^{\pi} \sin^3\phi d\phi\right)\left(\int_{\rho=0}^{1} \rho^4 d\rho\right)$$

$$= \left(\int_{\theta=0}^{2\pi} \left(\frac{1}{2} + \frac{1}{2}cos2\theta\right) d\theta\right)\left(\int_{\phi=0}^{\pi} sin\phi(1 - \cos^2\phi)d\phi\right)\left(\frac{1}{5}\rho^5 \Big|_0^1\right)$$

$$= (\pi)\left(-cos\phi + \frac{1}{3}\cos^3\phi\right) \Big|_0^{\pi}\left(\frac{1}{5}\right) = \frac{4\pi}{15}.$$

Method #2: Upper hemisphere is $z = \sqrt{1 - x^2 - y^2}$

Lower hemisphere is $z = -\sqrt{1 - x^2 - y^2}$



$$\iiint_W x^2\, dV = \iint_{x^2+y^2\leq 1} \int_{z=-\sqrt{1-x^2-y^2}}^{z=\sqrt{1-x^2-y^2}} x^2 dzdydx$$

$$= \iint_{x^2+y^2\le 1} x^2 z \ \Big|_{z=-\sqrt{1-x^2-y^2}}^{z=\sqrt{1-x^2-y^2}} dydx$$

$$= \iint_{x^2+y^2\le 1} 2x^2\sqrt{1-x^2-y^2}\,dydx$$

Now change to polar coordinates:

$$= \int_{\theta=0}^{2\pi}\int_{r=0}^{1} 2r^2\cos^2\theta(1-r^2)^{\frac{1}{2}} r\,dr\,d\theta$$

$$= \left(\int_{\theta=0}^{2\pi}\cos^2\theta d\theta\right)\left(\int_{r=0}^{r=1} 2r^3(1-r^2)^{\frac{1}{2}}dr\right).$$

For the first integral substitute $\cos^2\theta = \frac{1}{2}+\frac{1}{2}cos2\theta$.

For the second integral let $u = 1-r^2, \quad -du = 2rdr$, and $r^2 = 1-u$.

$$= \left(\int_{\theta=0}^{2\pi}\left(\frac{1}{2}+\frac{1}{2}cos2\theta\right)d\theta\right)\left(\int_{u=1}^{u=0} -(1-u)u^{\frac{1}{2}}du\right)$$

$$= (\pi)\left(\int_{u=1}^{u=0}\left(-u^{\frac{1}{2}}+u^{\frac{3}{2}}\right)du = (\pi)\left(\frac{2}{3}-\frac{2}{5}\right) = \frac{4\pi}{15}\,.$$