Conservative Vector Fields

Def. If $\vec{F} = \nabla f$, where f is a C^1 function from $\mathbb{R}^3$ (or $\mathbb{R}^n$) to $\mathbb{R}$, we say $\vec{F}$ is a **Gradient** vector field or a **Conservative** vector field.

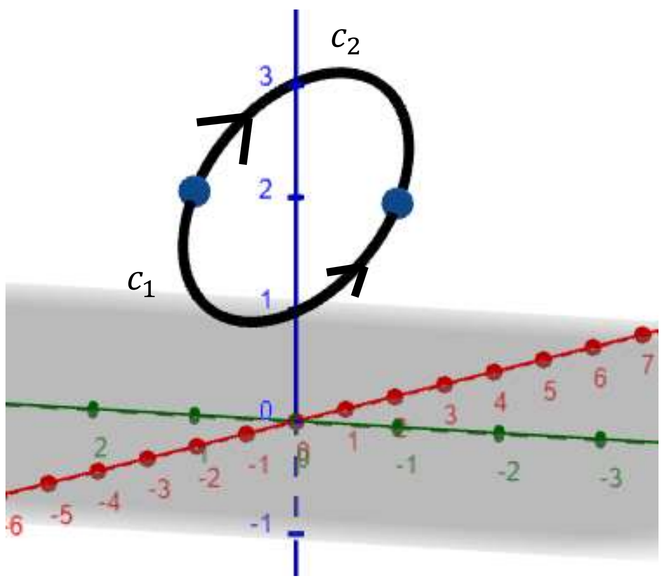
Recall that if $\vec{F}$ is a gradient vector field, ie $\vec{F} = \nabla f$, then:

$$\int_c \vec{F} \cdot d\vec{s} = \int_c \nabla f \cdot d\vec{s} = f\big(\vec{c}(b)\big) - f\big(\vec{c}(a)\big);$$

Where the C^1 path c begins at $\vec{c}(a)$ and ends at $\vec{c}(b)$. Thus, in this case, the value of the line integral $\int_c \vec{F} \cdot d\vec{s}$ depends only on the endpoints of the path c and NOT on the C^1 path itself. In this case we say the line integral is path independent.

Thus for any 2, C^1 paths, c_1 and c_2, which start and end at the same points we have:

$$\int_{c_1} \vec{F} \cdot d\vec{s} = \int_{c_2} \vec{F} \cdot d\vec{s} \;; \quad \text{if} \quad \vec{F} = \nabla f.$$



Notice that this means that if $\vec{F}$ is conservative (i.e., a gradient vector field), then

$$\int_c \vec{F} \cdot d\vec{s} = 0;$$

If c is an oriented, simple, closed curve.

Recall also that we saw that if $\vec{F} = \nabla f$, Then $\nabla \times \vec{F} = 0$.

This leads us to the following theorem.

Theorem (Conservative Vector Field): Let $\vec{F}$ be a C^1 vector field defined on $\mathbb{R}^3$, except possibly for a finite number of points. The following conditions on $\vec{F}$ are all equivalent.

1. For any oriented simple closed curve c, $\int_c \vec{F} \cdot d\vec{s} = 0$.

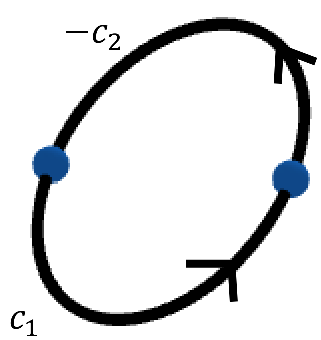
2. For any 2 oriented simple curves c_1 and c_2, which start and end at the same points, $\int_{c_1} \vec{F} \cdot d\vec{s} = \int_{c_2} \vec{F} \cdot d\vec{s}$.

3. $\vec{F}$ is the gradient of some function f; ie $\vec{F} = \nabla f$ (and if $\vec{F}$ has one or more exceptional points where if fails to be defined, f is also undefined).

4. $\nabla \times \vec{F} = 0$.

Proof: Show that $1 \Rightarrow 2 \Rightarrow 3 \Rightarrow 4 \Rightarrow 1$.

$1 \Rightarrow 2$: Since c_1 and c_2 are oriented simple curves which start and end at the same points, $c = c_1 + (-c_2)$ is an oriented closed curve. If it's not a simple closed curve break it up into a sum of simple closed curves.



By “1” we know that

$$0 = \int_c \vec{F} \cdot d\vec{s} = \int_{c_1} \vec{F} \cdot d\vec{s} - \int_{c_2} \vec{F} \cdot d\vec{s}$$

So we have: $\int_{c_1} \vec{F} \cdot d\vec{s} = \int_{c_2} \vec{F} \cdot d\vec{s}$.

2⇒ 3: Suppose $\vec{F}(x,y,z) =< F_1(x,y,z), F_2(x,y,z), F_3(x,y,z) >$.

Let’s show that assuming $\int_{c_1} \vec{F} \cdot d\vec{s} = \int_{c_2} \vec{F} \cdot d\vec{s}$ for any two oriented simple curves which start and end at the same point that we can find a function $f(x,y,z)$, such that $\vec{F} = \nabla f$.

Let c be any oriented simple curve joining the points $(0,0,0)$ and (x,y,z). Let’s define a function, $f(x,y,z)$ by

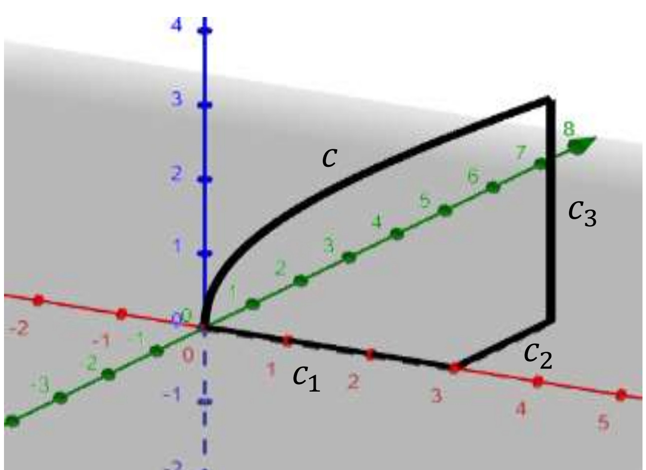
$$f(x,y,z) = \int_c \vec{F} \cdot ds = \int_c F_1 dx + F_2 dy + F_3 dz\,.$$

By assumption this function doesn’t depend on which oriented simple curve c that we use. For any fixed point (x,y,z) let’s choose c to be defined as the union of three line segments given by;

$$\vec{c}_1(t) =< t,0,0 >; \quad 0 \le t \le x; \quad \frac{dx}{dt} = 1, \quad \frac{dy}{dt} = \frac{dz}{dt} = 0$$

$$\vec{c}_2(t) =< x,t,0 >; \quad 0 \le t \le y \quad \frac{dy}{dt} = 1, \quad \frac{dx}{dt} = \frac{dz}{dt} = 0$$

$$\vec{c}_3(t) =< x,y,t >; \quad 0 \le t \le z \quad \frac{dz}{dt} = 1, \quad \frac{dx}{dt} = \frac{dy}{dt} = 0.$$



$$f(x,y,z) = \int_c F_1 dx + F_2 dy + F_3 dz$$

$$= \int_0^x F_1(t,0,0)dt + \int_0^y F_2(x,t,0)dt + \int_0^z F_3(x,y,t)dt$$

Notice that $\frac{\partial f}{\partial z} = F_3(x,y,z)$.

Now repeat the argument with a path going from $(0,0,0)$ to (x,y,z) by line segments going from $(0,0,0)$ to $(0,y,0)$ to $(0,y,z)$ to (x,y,z). This will show that $\frac{\partial f}{\partial x} = F_1(x,y,z)$.

Now repeat the argument with a path going from $(0,0,0)$ to (x,y,z) by line segments going from $(0,0,0)$ to $(x,0,0)$ to $(x,0,z)$ to (x,y,z). This will show that $\frac{\partial f}{\partial y} = F_2(x,y,z)$.

Hence, $\vec{F} = \nabla f$.

3⇒ 4: We have already seen that if $\vec{F} = \nabla f$, then $\nabla \times \vec{F} = curl(grad(f)) = 0$.

4⇒ 1: Let S be a surface with $\partial S = c$, then by Stokes' theorem:

$$\int_c \vec{F} \cdot d\vec{s} = \iint_S (\nabla \times \vec{F}) \cdot d\vec{S} = \iint_S \vec{0} \cdot d\vec{S} = 0.$$

Def. If $\vec{V}$ is the velocity vector field of a fluid and c is a closed simple curve then

$\int_c \vec{V} \cdot d\vec{s}$ is called the **Circulation of $\vec{V}$ around c**.

Since $\int_c \vec{V} \cdot d\vec{s} = \int_c \vec{V} \cdot \vec{T} ds$, where $\vec{T}$ is the unit tangent vector to the curve c, circulation measures the net amount of turning of the fluid around the curve.

Recall that if $curl(\vec{F}) = 0$, we say that $\vec{F}$ is irrotational. By our theorem, $\vec{F}$ is irrotational if $\vec{F}$ has no circulation, i.e. $\int_c \vec{F} \cdot d\vec{s} = 0$ for all oriented simple closed curves.

Also, by the theorem, a vector field $\vec{F}$ is irrotational, if and only if $\vec{F} = \nabla f$.

The function f is called the **Potential Function** for $\vec{F}$.

Ex. Let $\vec{F} =< z, (y + ze^{yz}), (x + ye^{yz}) >$. Show $\vec{F}$ is irrotational and find all scalar potential functions for $\vec{F}$.

$$\nabla \times \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ z & y + ze^{yz} & x + ye^{yz} \end{vmatrix} = \vec{0}.$$

By our theorem, since $\nabla \times \vec{F} = 0$, we know that $\int_c \vec{F} \cdot d\vec{s} = 0$ for all oriented simple closed curves. Thus, $\vec{F}$ is irrotational.

If we know that $\vec{F} = \nabla f$, how do we find all of the potential functions f? Let's illustrate the method through an example.

Suppose $\vec{F} =< z, (y + ze^{yz}), (x + ye^{yz}) >$. One can check that $\nabla \times \vec{F} = 0$ and hence $\vec{F}$ is a gradient vector field.

$$\nabla f =< \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} >= \vec{F} =< z, (y + ze^{yz}), (x + ye^{yz}) >.$$

$$\frac{\partial f}{\partial x} = z \qquad \frac{\partial f}{\partial y} = y + ze^{yz} \qquad \frac{\partial f}{\partial z} = x + ye^{yz}.$$

Start by taking the "easiest" equation to integrate with respect to the relevant variable: ie, either integrate the first equation with respect to x, the second with respect to y, or the third with respect to z. In this case it looks like integrating the first with respect to x is the easiest.

$f(x, y, z) = \int z dx = xz + h(y, z)$.

That is, if you differentiate $xz + h(y, z)$ with respect to x you get z.

Now we need to find $h(y, z)$.

Since we integrated with respect to x, now differentiate our $f(x, y, z)$ with respect to either y or z. Let's take y first.

$\frac{\partial f}{\partial y} = 0 + \frac{\partial h(y,z)}{\partial y}$; but from our 3 original differential equations that we know:

$\frac{\partial f}{\partial y} = y + ze^{yz} = 0 + \frac{\partial h(y,z)}{\partial y} = \frac{\partial h(y,z)}{\partial y}$; So we now have:

$h(y, z) = \int (y + ze^{yz}) dy = \frac{y^2}{2} + e^{yz} + g(z)$; So that means:

$f(x, y, z) = xz + \frac{y^2}{2} + e^{yz} + g(z)$.

We can use the third equation, $\frac{\partial f}{\partial z} = x + ye^{yz}$, to find $g(z)$.

$$\frac{\partial f}{\partial z} = x + ye^{yz} = \frac{\partial}{\partial z}\left(xz + \frac{y^2}{2} + e^{yz} + g(z) \right) = x + ye^{yz} + g'(z)$$

This means that $g'(z) = 0$ and $g(z) = c$. So finally we have:

$$f(x, y, z) = xz + \frac{y^2}{2} + e^{yz} + c.$$

Ex. Evaluate $\int_c \vec{F} \cdot d\vec{s}$, where $\vec{F} =< z, (y + ze^{yz}), (x + ye^{yz}) >$, and

$\vec{c}(t) =< t^2 \sin\left(\frac{\pi}{2} t\right), te^{\sqrt{t-1}}, t^6 >, \quad 0 \le t \le 1.$

Method 1: From the previous example we know that $\vec{F}$ is conservative and that the potential functions for $\vec{F}$ are $f(x, y, z) = \frac{y^2}{2} + (xz + e^{yz}) + C$.

Since $\vec{F}$ is conservative we know that

$$\int_c \vec{F} \cdot d\vec{s} = \int_c \nabla f \cdot d\vec{s} = f(\vec{c}(b)) - f(\vec{c}(a));$$

where the C^1 path c begins at $\vec{c}(a)$ and ends at $\vec{c}(b)$.

$\vec{c}(0) =< 0,0,0 > \quad and \quad \vec{c}(1) =< 1,1,1 >$ and

$f(1,1,1) = 1 + \frac{1}{2} + e + C = \frac{3}{2} + e + C$

$f(0,0,0) = 0 + 0 + 1 + C = 1 + C$

$\int_c \vec{F} \cdot d\vec{s} = f(1,1,1) - f(0,0,0) = \frac{1}{2} + e.$

Method 2: From the previous example we know that $\vec{F}$ is conservative. Thus $\int_c \vec{F}\cdot d\vec{s}$ will have the same value for ANY C^1 path c which has the same endpoints as $\vec{c}(t) =< t^2 \sin\left(\frac{\pi}{2}t\right), te^{\sqrt{t-1}}, t^6 >, \quad 0 \le t \le 1$. So let's choose a path to integrate along which has the same endpoints, but is much simpler.

Notice that:

$\vec{c}(0) =< 0,0,0 > \quad and \quad \vec{c}(1) =< 1,1,1 >.$

The simplest path is a line segment between $< 0,0,0 >$ and $< 1,1,1 >$.

So let $\vec{c_1}(t) =< t, t, t >, \quad 0 \le t \le 1. \qquad \vec{c_1}'(t) =< 1, 1, 1 >.$

$\vec{F}(\vec{c}(t)) =< t, t + te^{t^2}, t + e^{t^2} >.$

$$\int_c \vec{F}\cdot d\vec{s} = \int_{c_1} \vec{F}\cdot d\vec{s} = \int_{t=0}^{t=1} < t, t + te^{t^2}, t + te^{t^2} > \cdot < 1, 1, 1 > \mathrm{dt}$$

$$= \int_0^1 (t + t + \mathrm{t}e^{t^2} + t + te^{t^2})dt = \int_0^1 (3t + 2te^{t^2})dt = \frac{1}{2} + e.$$

Ex. Suppose an object with mass M is at the origin in $\mathbb{R}^3$ and exerts a force on a second object with mass m located at $\vec{r} =< x, y, z >$ with a magnitude of $\frac{GmM}{|\vec{r}|^2}$, where G is a gravitational constant and the direction is toward the origin. We can thus write the force field as:

$$\vec{F}(x, y, z) = \left(\frac{GmM}{|\vec{r}|^2}\right)\left(-\frac{\vec{r}}{|\vec{r}|}\right) = -\left(\frac{GmM}{|\vec{r}|^3}\right)\vec{r}.$$

Show $\vec{F}$ is irrotational (ie $\nabla \times \vec{F} = 0$) and find all of the potential functions for $\vec{F}$.

So we first have to show that $\nabla \times \vec{F} = 0$.

Since $\vec{r} =< x, y, z >$, $\quad |\vec{r}| = \sqrt{x^2 + y^2 + z^2}$.

$\nabla \times \vec{F} = \nabla \times \left(-\left(\frac{GmM}{|\vec{r}|^3}\right)\vec{r}\right) = (-GmM)(\nabla \times \frac{\vec{r}}{|\vec{r}|^3})$; where:

$$\frac{\vec{r}}{|\vec{r}|^3} =< \frac{\mathrm{x}}{(x^2+y^2+z^2)^{\frac{3}{2}}}, \frac{\mathrm{y}}{(x^2+y^2+z^2)^{\frac{3}{2}}}, \frac{\mathrm{z}}{(x^2+y^2+z^2)^{\frac{3}{2}}} >$$

$$\left(\nabla \times \frac{\vec{r}}{|\vec{r}|^3}\right) = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{\mathrm{x}}{(x^2+y^2+z^2)^{\frac{3}{2}}} & \frac{\mathrm{y}}{(x^2+y^2+z^2)^{\frac{3}{2}}} & \frac{\mathrm{z}}{(x^2+y^2+z^2)^{\frac{3}{2}}} \end{vmatrix} = \vec{0}.$$

So $\nabla \times \vec{F} = 0$, thus we know that $\vec{F}$ is irrotational and that there is a function, $f(x, y, z)$ such that $\vec{F} = \nabla f$.

Since $\nabla f = \vec{F} = (-GmM) < \frac{\mathrm{x}}{(x^2+y^2+z^2)^{\frac{3}{2}}}, \frac{\mathrm{y}}{(x^2+y^2+z^2)^{\frac{3}{2}}}, \frac{\mathrm{z}}{(x^2+y^2+z^2)^{\frac{3}{2}}} >$

$$\frac{\partial f}{\partial x} = \frac{-\mathrm{GmMx}}{(x^2+y^2+z^2)^{\frac{3}{2}}} \qquad \frac{\partial f}{\partial y} = \frac{-\mathrm{GmMy}}{(x^2+y^2+z^2)^{\frac{3}{2}}} \qquad \frac{\partial f}{\partial z} = \frac{-\mathrm{GmMz}}{(x^2+y^2+z^2)^{\frac{3}{2}}}.$$

Choose any of the 3 differental equations to integrate with respect to the relevant variable (if you use $\frac{\partial f}{\partial x}$, integrate with respect to x, or $\frac{\partial f}{\partial y}$, integrate with respect to y, etc.). Let's integrate the first equation.

$$f(x,y,z) = \int \frac{-\mathrm{GmMx}}{(x^2+y^2+z^2)^{\frac{3}{2}}}\mathrm{dx} = -\mathrm{GmM}\int \frac{\mathrm{x}}{(x^2+y^2+z^2)^{\frac{3}{2}}}\mathrm{dx}$$

$$= \mathrm{GmM}(x^2+y^2+z^2)^{\frac{-1}{2}} + \mathrm{h(y,z)}.$$

$$\frac{\partial f}{\partial y} = \frac{-\mathrm{GmMy}}{(x^2+y^2+z^2)^{\frac{3}{2}}} = \frac{\partial}{\partial \mathrm{y}}(\mathrm{GmM}(x^2+y^2+z^2)^{\frac{-1}{2}} + \mathrm{h(y,z)})$$

$$= -GmM\left(\frac{\mathrm{y}}{(x^2+y^2+z^2)^{\frac{3}{2}}}\right) + \frac{\partial \mathrm{h}}{\partial \mathrm{y}}$$

So $\frac{\partial \mathrm{h}}{\partial \mathrm{y}} = 0$, and $\mathrm{h(y,z)} = \mathrm{h(z)}$.

So now we know that:

$f(x,y,z) = \mathrm{GmM}(x^2+y^2+z^2)^{\frac{-1}{2}} + \mathrm{h(z)}$. Now use the 3rd differential eq.

$$\frac{\partial f}{\partial z} = \frac{-\mathrm{GmMz}}{(x^2+y^2+z^2)^{\frac{3}{2}}} = \frac{\partial}{\partial \mathrm{z}}(\mathrm{GmM}(x^2+y^2+z^2)^{\frac{-1}{2}} + \mathrm{h(z)})$$

$$= -GmM\left(\frac{\mathrm{z}}{(x^2+y^2+z^2)^{\frac{3}{2}}}\right) + \mathrm{h}'(\mathrm{z})$$

So $\mathrm{h}'(\mathrm{z}) = 0$, and $\mathrm{h(z)} = \mathrm{c}$.

So finally all of the potential functions, $f(x,y,z)$, called the gravitational potential energy are given by:

$$f(x,y,z) = \mathrm{GmM}(x^2+y^2+z^2)^{\frac{-1}{2}} + \mathrm{C}.$$

Gravitational potential energy goes to 0 as $|\vec{r}|$ goes to ∞, thus $C = 0$ and

$$f(x, y, z) = \mathrm{GmM}(x^2 + y^2 + z^2)^{\frac{-1}{2}}.$$

We can also write this as:

$$f(x, y, z) = \mathrm{GmM}(x^2 + y^2 + z^2)^{\frac{-1}{2}} = \frac{\mathrm{GmM}}{|\vec{r}|}$$

<u>Conservative Vector Fields in the Plane</u>

The Conservative Vector Field Theorem also holds in the plane, however, we need to require that there are no exceptional points (points where the vector field doesn't exist).

For a vector field $\vec{F}$ in $\mathbb{R}^2$, we have:

$$\nabla \times \vec{F} = \begin{vmatrix} \vec{\imath} & \vec{\jmath} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_1 & F_2 & 0 \end{vmatrix} = \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right)\vec{k} = \vec{0}$$ implies that

$\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} = 0;$ or $\frac{\partial F_2}{\partial x} = \frac{\partial F_1}{\partial y}$.

Corrolary to Conservative Vector Field Theorem: If $\vec{F}$ is a C^1 vector field on $\mathbb{R}^2$ of the form $\vec{F} = F_1(x, y)\vec{\imath} + F_2(x, y)\vec{\jmath}$ that satisfies $\frac{\partial F_2}{\partial x} = \frac{\partial F_1}{\partial y}$, then $\nabla \times \vec{F} = 0$, and $\vec{F} = \nabla f$ for some function $f(x, y)$ in $\mathbb{R}^2$.

Ex. Determine if the following vector fields are gradient fields (ie, conservative fields). If it is, find a potential function f, ie f such that $\vec{F} = \nabla f$.

a. $\vec{F}(x,y) = (\sin(xy))\vec{\imath} + y^2\vec{\jmath}$

b. $\vec{G}(x,y) = < 2xcosy, -x^2siny >$

a. We first need to check to see if $\frac{\partial F_2}{\partial x} = \frac{\partial F_1}{\partial y}$

$\frac{\partial F_2}{\partial x} = 0$ and $\frac{\partial F_1}{\partial y} = xcos(xy)$; so $\frac{\partial F_2}{\partial x} \neq \frac{\partial F_1}{\partial y}$ and $\vec{F} \neq \nabla f$.

b. $\frac{\partial G_2}{\partial x} = -2xsiny$ and $\frac{\partial G_1}{\partial y} = -2xsiny$; so $\frac{\partial G_2}{\partial x} = \frac{\partial G_1}{\partial y}$ and $\vec{G} = \nabla f$.

Since $\vec{G} = \nabla f$, we have:

$$\frac{\partial f}{\partial x} = 2xcosy \quad \text{and} \quad \frac{\partial f}{\partial y} = -x^2siny.$$

As in the case when we had 3 differential equations to solve, choose the easiest one to integrate. In this case, either one will do:

$$f(x,y) = \int(2xcosy)dx = x^2cosy + h(y).$$

Now differentiate this function with respect to the other varable (in this case y).

$$\frac{\partial f}{\partial y} = -x^2siny + h^{'}(y).$$

But we already have an expression for $\frac{\partial f}{\partial y}$ from our differential equations:

$$\frac{\partial f}{\partial y} = -x^2siny.$$

Now set these two expressions for $\frac{\partial f}{\partial y}$ equal and solve for $h'(y)$.

$\frac{\partial f}{\partial y} = -x^2 siny + h^{'}(y)$= $-x^2 siny$.

So, $h'(y)$=0, and $h(y) = C$. So we can finally say that:

$f(x, y) = x^2 cosy + C$.

Ex. Let c: $[2,5] \rightarrow \mathbb{R}^2$, by $x = e^{(t-2)}$, $y = \cos(\pi t)$. Find $\int_c \vec{F} \cdot d\vec{s}$, where

$\vec{F}(x, y) = (2\text{xcosy})\vec{\imath} - (x^2 siny)\vec{\jmath}$.

We just showed that $\vec{F}(x, y)$ is a gradient vector field and:

$\nabla(x^2 cosy + C) = \vec{F}(x, y)$.

We also know for Gradient vector fields:

$\int_c \vec{F} \cdot d\vec{s} = \int_c \nabla f \cdot d\vec{s} = f\big(\vec{c}(b)\big) - f\big(\vec{c}(a)\big)$.

In this case, $a = 2$ and $b = 5$ so we have:

$\vec{c}(5) = \left(e^{(5-2)}, \cos(5\pi)\right) = (e^3, -1)$ and

$\vec{c}(2) = \left(e^{(2-2)}, \cos(2\pi)\right) = (1,1)$.

$\int_c \vec{F} \cdot d\vec{s} = f(e^3, -1) - f(1,1) = (e^3)^2(\cos(-1)) - (1)^2(\cos(1))$

$= e^6(\cos(-1)) - \cos(1) = (e^6 - 1)\cos(1)$.

We know that $Div\left(Curl(\vec{G})\right) = 0$, if $\vec{G}$ is a C^2 vector field. But, if $Div(\vec{F}) = 0$, does this mean that $\vec{F} = Curl(\vec{G}\,)$, for some vector field $\vec{G}$?

Theorem: If $\vec{F}$ is a C^1 vector field on $\mathbb{R}^3$ with $Div(\vec{F}) = 0,$ then there is a C^1 vector field $\vec{G}$ such that $\vec{F} = Curl(\vec{G}\,)$.

Proof: If $\vec{G} = G_1\vec{\imath} + G_2\vec{\jmath} + G_3\vec{k}$; and $\vec{F} = F_1\vec{\imath} + F_2\vec{\jmath} + F_3\vec{k}$; Let:

$$G_1 = \int_0^z F_2(x,y,t)dt - \int_0^y F_3(x,t,0)dt$$

$$G_2 = -\int_0^z F_1(x,y,t)dt$$

$$G_3 = 0.$$

Now let's show that $Curl(\vec{G}\,) = \vec{F}$.

$$\nabla \times \vec{G} = \begin{vmatrix} \vec{\imath} & \vec{\jmath} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ G_1 & G_2 & G_3 \end{vmatrix} = \begin{vmatrix} \vec{\imath} & \vec{\jmath} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ G_1 & G_2 & 0 \end{vmatrix} = -\frac{\partial G_2}{\partial z}\vec{\imath} + \frac{\partial G_1}{\partial z}\vec{\jmath} + \left(\frac{\partial G_2}{\partial x} - \frac{\partial G_1}{\partial y}\right)\vec{k}.$$

$G_2 = -\int_0^z F_1(x,y,t)dt;$ so

$-\frac{\partial G_2}{\partial z} = F_1(x,y,z)$ (by the second fundamental theorem of calculus)

$G_1 = \int_0^z F_2(x,y,t)dt - \int_0^y F_3(x,t,0)dt$; so

$$\frac{\partial G_1}{\partial z} = F_2(x,y,z) - 0 = F_2(x,y,z).$$

$$\frac{\partial G_2}{\partial x} - \frac{\partial G_1}{\partial y} = -\int_0^z \frac{\partial F_1}{\partial x}(x,y,t)dt - \int_0^z \frac{\partial F_2}{\partial y}(x,y,t)dt + \frac{\partial}{\partial y}\int_0^y F_3(x,t,0)dt.$$

$$= -\int_0^z \frac{\partial F_1}{\partial x}(x,y,t)dt - \int_0^z \frac{\partial F_2}{\partial y}(x,y,t)dt + F_3(x,y,0)$$

$$\frac{\partial G_2}{\partial x} - \frac{\partial G_1}{\partial y} = \int_0^z -\frac{\partial F_1}{\partial x}(x,y,t) - \frac{\partial F_2}{\partial y}(x,y,t)dt + F_3(x,y,0)\ .$$

Now since $Div(\vec{F}) = 0$, we have $\frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z} = 0$.

Or we can write: $\frac{\partial F_3}{\partial z} = -\frac{\partial F_1}{\partial x} - \frac{\partial F_2}{\partial y}$. Substituting this in we get:

$$\frac{\partial G_2}{\partial x} - \frac{\partial G_1}{\partial y} = \int_0^z \frac{\partial F_3}{\partial z}(x,y,t)\, dt + F_3(x,y,0);$$ Now using the FTC

$$\frac{\partial G_2}{\partial x} - \frac{\partial G_1}{\partial y} = (F_3(x,y,z) - F_3(x,y,0)) + F_3(x,y,0)$$

$$\frac{\partial G_2}{\partial x} - \frac{\partial G_1}{\partial y} = F_3(x,y,z);$$

So we have:

$$\nabla \times \vec{G} = -\frac{\partial G_2}{\partial z}\vec{\imath} + \frac{\partial G_1}{\partial z}\vec{\jmath} + \left(\frac{\partial G_2}{\partial x} - \frac{\partial G_1}{\partial y}\right)\vec{k} = F_1\vec{\imath} + F_2\vec{\jmath} + F_3\vec{k} = \vec{F}.$$

Ex. Which of the following vector fields is the curl of another vector field? Find the vector field if it exists.

a. $\vec{F}(x,y,z) = ze^{x}\vec{\imath} + ye^{x}\vec{\jmath} + z\vec{k}$

$b.$ $\vec{F}(x,y,z) = x^{2}\vec{\imath} + (z+xy)\vec{\jmath} - 3xz\vec{k}$

a. A vector field $\vec{F}$, is the curl of another vector field $\vec{G}$, if and only if, $Div\vec{F} = 0.$

$Div\vec{F} = ze^{x} + e^{x} + 1 \neq 0;$ so $\vec{F} \neq Curl\vec{G}$.

b. $Div\vec{F} = 2x + x - 3x = 0;$ so $\vec{F} = Curl\vec{G}$, for some vector field $\vec{G}$.

We know from the proof of the last theorem that $\vec{G} = G_1\vec{\imath} + G_2\vec{\jmath} + G_3\vec{k}$, where:

$G_1 = \int_0^z F_2(x,y,t)dt - \int_0^y F_3(x,t,0)dt$

$G_2 = -\int_0^z F_1(x,y,t)dt$

$G_3 = 0$

$F_1 = x^2,$ $F_2 = z + xy,$ $F_3 = -3xz$ so we get:

$G_1 = \int_0^z F_2(x,y,t)dt - \int_0^y F_3(x,t,0)dt = \int_0^z (t+xy)dt - \int_0^y 0dt$

$= \frac{t^2}{2} + txy|_{t=0}^{t=z}$,

$= \frac{z^2}{2} + zxy.$

$$G_2 = -\int_0^z F_1(x,y,t)dt = -\int_0^z x^2 dt = -x^2 t|_{t=0}^{t=z} \ ,$$

$$= -x^2 z.$$

Thus we have:

$$\vec{G} = G_1\vec{\imath} + G_2\vec{\jmath} + G_3\vec{k} = \left(\frac{z^2}{2} + zxy\right)\vec{\imath} - (x^2 z)\vec{\jmath}$$

You can check that $\nabla \times \vec{G} = \vec{F}$.