Green's Theorem

Green's theorem is a generalization of the fundamental theorem of Calculus:

$$\int_a^b f'(x)dx = f(b) - f(a).$$

Notice that the FTC relates the definite integral of $f'(x)$ over a line segment, $[a, b]$, with the value of the function $f(x)$ at the endpoints of that interval (i.e. at a and b). Another way to think of the endpoints of an interval (i.e., a line segment) is as the boundary of that interval (much the way the unit circle is the boundary of the unit disk, $x^2 + y^2 \leq 1$). So the FTC relates the definite integral of $f'(x)$ over a line segment with the value of the function $f(x)$ on the boundary of the interval.

Green's theorem relates a double integral of a function (we will see later how this function is similar to the "derivative" of another function) over an elementary region in $\mathbb{R}^2$ with a line integral around the boundary of that region.

Green's Theorem: Let D be a simple region and let c be its boundary. Suppose

$P: D \to \mathbb{R}$ and $Q: D \to \mathbb{R}$ are both C^1 functions then

$$\int_c P(x,y)dx + Q(x,y)dy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)dxdy.$$

Note: We denote the "boundary of D" by ∂D, so we could rewrite Green's thm as:

$$\int_{\partial D} P(x,y)dx + Q(x,y)dy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)dxdy.$$

Green's theorem is useful because sometimes one side of Green's theorem is much easier to evaluate than the other side.

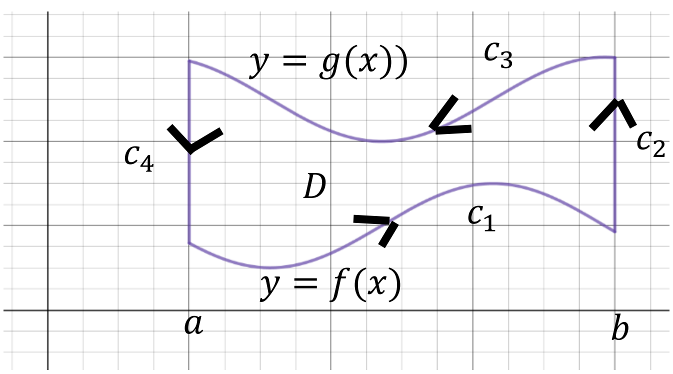
We can prove Green's theorem by proving the following two lemmas.

Lemma 1: Let D be a y-simple region and let $c = \partial D$. Suppose $P: D \to \mathbb{R}$ is of class C^1 then: $\int_c P(x,y)dx = -\iint_D \frac{\partial P}{\partial y} dxdy$.

Lemma 2: Let D be an x-simple region and let $c = \partial D$. Suppose $Q: D \to \mathbb{R}$ is of class C^1 then: $\int_c Q(x,y)dx = \iint_D \frac{\partial Q}{\partial x} dxdy$.

Adding the conclusions to lemma 1 and lemma 2 gives Green's theorem.

To prove lemma 1 let's assume that D is bounded below by the curve $y = f(x)$, above by the curve $y = g(x)$ and on the sides by $x = a$ and $x = b$.
$\partial D = c = c_1 + c_2 - c_3 - c_4$ (oriented counterclockwise) where



$\vec{c}_1(t) =< t, f(t) >: \quad a \le t \le b$

$\vec{c}_2(t) =< b, t >; \quad f(b) \le t \le g(b)$

$\vec{c}_3(t) =< t, g(t) >; \quad a \le t \le b$

$\vec{c}_4(t) =< a, t >; \quad f(a) \le t \le g(a)$.

By Fubini's theorem we can say:

$$\iint_D \frac{\partial P}{\partial y} dxdy = \int_{x=a}^{x=b} [\int_{y=f(x)}^{y=g(x)} \frac{\partial P}{\partial y} dy] dx.$$

Now by the fundamental theorem of Calculus we have:

$$\iint_D \frac{\partial P}{\partial y} dxdy = \int_{x=a}^{x=b} [P(x, g(x)) - P(x, f(x))] dx.$$

Now let's evaluate the line integral around $\partial D = c$:

$$\int_c P(x,y)dx = \int_{t=a}^{t=b} P(t, f(t))dt - \int_{t=a}^{t=b} P(t, g(t))dt.$$

Notice that the line integrals along c_2 and c_4 are both 0 because $\frac{dx}{dt} = 0$.

If we rewrite this line integral around c, replacing x for t we get:

$$\int_c P(x,y)dx = \int_{x=a}^{x=b} \left[P\big(x, f(x)\big) - P\big(x, g(x)\big)\right] dx$$

$$= -\int_{x=a}^{x=b} [P\big(x, g(x)\big) - P\big(x, f(x)\big)] dx.$$

Now notice from our earlier calculation for $\iint_D \frac{\partial P}{\partial y} dxdy$ we have:

$$\int_c P(x,y)dx = -\iint_D \frac{\partial P}{\partial y} dxdy.$$

A similar argument also proves lemma 2.

Ex. Evaluate $\int_c \left(3y - e^{sinx}\right)dx + (7x + \sqrt{y^4 + 1})dy$; where c is the circle $x^2 + y^2 = 9$ oriented counterclockwise.

Suppose we just try to evaluate this line integral directly.

$x = 3\text{cost}$, $\quad$ $\text{y} = 3\text{sint}$; $\quad$ $\text{dx} = (-3\text{sint})\text{dt}$, $\quad$ $\text{dy} = (3\text{cost})\text{dt}$.

So the line integral becomes:

$$\int_0^{2\pi} \left(9sint - e^{\sin(3cost)}\right)(-3sint)dt + (21cost + \sqrt{81sin^4t + 1})(3cost)dt$$

Ugh!!!

Now use Green's theorem:

$P(x,y) = 3y - e^{sinx}$ $\qquad$ $Q(x,y) = 7x + \sqrt{y^4+1}$

$\frac{\partial P}{\partial y} = 3$ $\qquad$ $\frac{\partial Q}{\partial x} = 7$

$$\int_c (3y - e^{sinx})dx + (7x + \sqrt{y^4+1})dy = \iint_D (\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y})dxdy$$
$$= \iint_D (7-3)dxdy = \iint_D (4)dxdy$$

where D is the disk about the origin of radius 3.

$= 4\iint_D\ dxdy$= 4(area of circle of radius 3)$= 4(9\pi) = 36\pi.$

Sometimes the line integral is easier than the double integral. For example, if you knew that $P(x,y) = Q(x,y) = 0$ on the curve c (but not necessarily away from c). In this case:

$$\iint_D (\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y})dxdy = \int_c P(x,y)dx + Q(x,y)dy = 0.$$

Ex. Verify Green's theorem for $P(x,y) = y,\ \ Q(x,y) = -x,$ on the disk of radius 4.

$$\int_c P(x,y)dx + Q(x,y)dy = \iint_D (\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y})dxdy$$

$\frac{\partial P}{\partial y} = 1, \qquad \frac{\partial Q}{\partial x} = -1$

The boundary of a disk of radius 4 is the circle of radius 4:

$x = 4cost, \qquad y = 4sint, \qquad dx = (-4sint)dt, \qquad dy = (4cost)dt.$

$\int_c P(x,y)dx + Q(x,y)dy =$

$$\int_c ydx - xdy = \int_0^{2\pi} (4sint)(-4sint)dt - (4cost)(4cost)dt$$

$$= \int_0^{2\pi} (-16sin^2 t - 16cos^2 t)dt = \int_0^{2\pi} -16dt = -32\pi.$$

$$\iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)dxdy = \iint_D (-1-1)dxdy = \iint_D -2dxdy$$

$$= -2(area\ of\ disk) = -2(16\pi) = -32\pi.$$

Ex. Verify Green's Theorem for $P(x,y) = y,\ Q(x,y) = -x$ on:

a. The rectangle $D = [-1,2] \times [1,3]$
b. The annulus $D = \{(x,y) \in \mathbb{R}^2 |\ 4 \le x^2 + y^2 \le 16\}$

$$\int_c P(x,y)dx + Q(x,y)dy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)dxdy$$

$$\frac{\partial P}{\partial y} = 1, \qquad \frac{\partial Q}{\partial x} = -1$$

To solve part "a" let's start with the LHS of the equation for Green's Theorem.

Here we have to parametrize the sides of the rectangle.

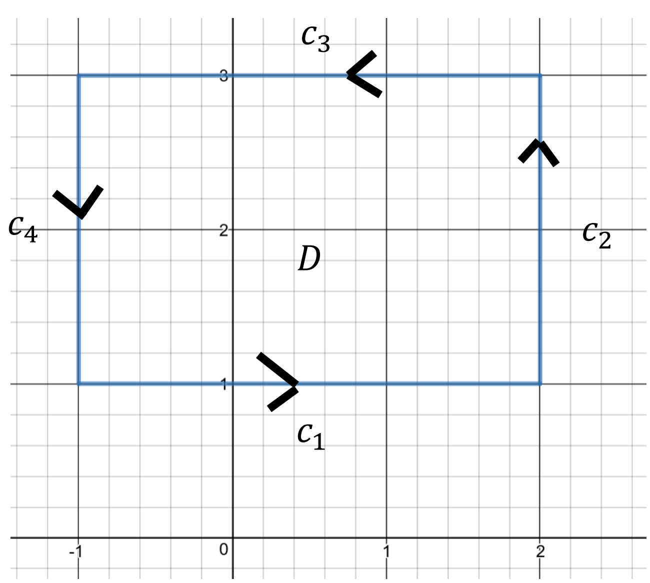
$\partial D = c_1 + c_2 + c_3 + c_4$; where

$\vec{c}_1(t) =< t, 1 > \quad -1 \le t \le 2$

$\vec{c}_1'(t) =< 1,0 >$

$\vec{c}_2(t) =< 2, t > \quad 1 \le t \le 3$

$\vec{c}_2'(t) =< 0,1 >$



$\vec{c}_3(t) =< -t, 3 > \quad -2 \le t \le 1$

$\vec{c}_3'(t) =< -1,0 >$

$\vec{c}_4(t) =< -1, -t > \; -3 \le t \le -1$

$\vec{c}_4'(t) =< 0, -1 >$

$\int_{\partial D} ydx - xdy =$

$$= \int_{c_1} ydx - xdy + \int_{c_2} ydx - xdy + \int_{c_3} ydx - xdy + \int_{c_4} ydx - xdy$$

$$= \int_{-1}^{2} 1dt + \int_{1}^{3} -2dt + \int_{-2}^{1} -3\,dt + \int_{-3}^{-1} -1dt$$

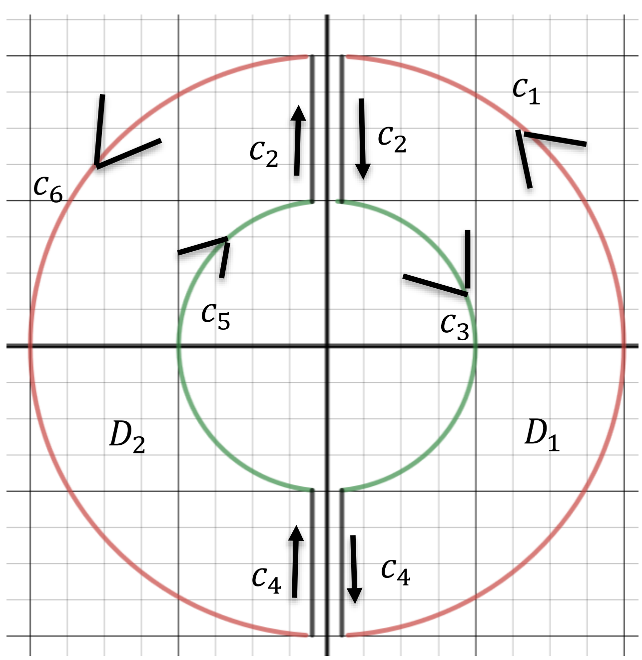
$$= 3 - 4 - 9 - 2 = -12.$$

For the RHS of Green's theorem we have:

$$\iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dxdy = \iint_D (-1-1)dxdy = -2\iint_D dxdy$$

$$= -2(area\ of\ rect.) = -2(3)(2) = -12.$$

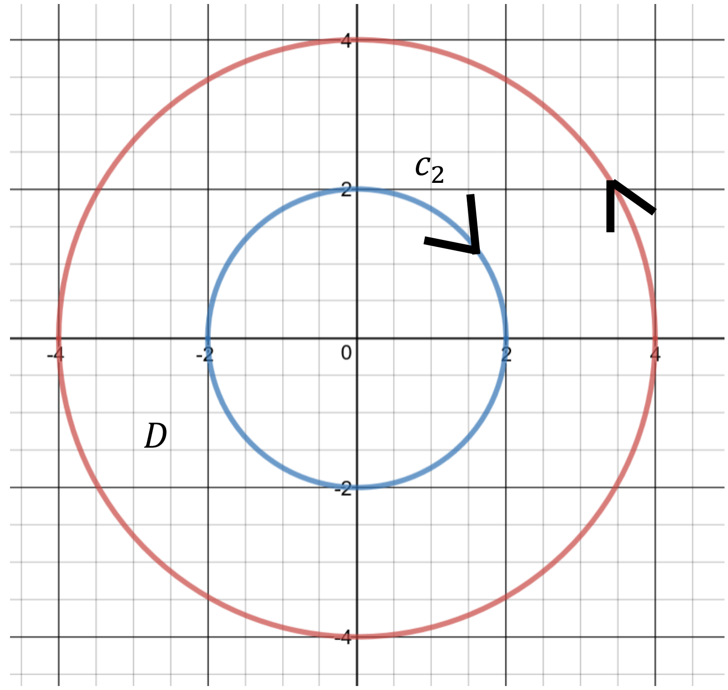
For part "b" notice that Green's theorem is actually true for any region that can be broken up into a finite union of simple regions.



Here the annulus, $D = D_1 \cup D_2$, is the union of two simple regions. Notice that the two boundary circles are oriented in opposite directions.

c_1

For part "b" we need to parametrize the 2 circles bounding the annulus. However, remember that they will be oriented in opposite directions.



$\vec{c}_1(t) =< 4cost, 4sint > \quad 0 \le t \le 2\pi$

(counterclockwise)

$\vec{c}_1'(t) =< -4sint, 4cost >$

$\vec{c}_2(t) =< 2cost, -2sint > \quad 0 \le t \le 2\pi$

(clockwise)

$\vec{c}_2'(t) =< -2sint, -2cost >$

$c = c_1 + c_2$

$$\int_c ydx - xdy = \int_{c_1} ydx - xdy + \int_{c_2} ydx - xdy$$

$$= \int_0^{2\pi} [4sint(-4sint) - 4cost(4cost)]dt$$

$$+ \int_0^{2\pi} [-2sint(-2sint) - 2cost(-2cost)]dt$$

$$= \int_0^{2\pi} -16 + \int_0^{2\pi} 4dt = -32\pi + 8\pi = -24\pi.$$

$$\iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)dxdy = \iint_D -2dxdy = -2(area\ of\ annulus)$$

$$= -2(\pi(4)^2 - \pi(2)^2) = -24\pi.$$

Theorem: Area of a Region. If c is a simple closed curve that bounds a region to which Green's theorem applies, then the area of the region D bounded by c=∂D is

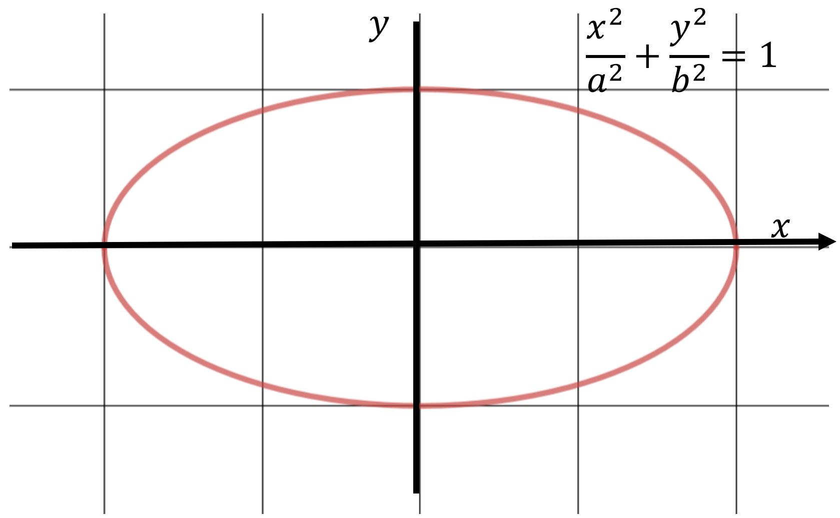
$$A = \frac{1}{2}\int_{\partial D}[(x)dy - (y)dx].$$

Proof: Let $P(x,y) = -y, \quad Q(x,y) = x$. Then by Green's theorem

$$\frac{1}{2}\int_{\partial D}[(x)dy - (y)dx] = \frac{1}{2}\iint_D \left(\frac{\partial x}{\partial x} - \left(-\frac{\partial y}{\partial y}\right)\right)dxdy = \frac{1}{2}\iint_D 2dxdy$$

$$= \text{Area of } D.$$

Ex. Find the area enclosed by the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.



We can parametrize the ellipse by:

$x = acost, \; y = bsint, \quad 0 \le t \le 2\pi.$

$dx = -(asint)dt, \quad dy = (bcost)dt$.

Area= $\frac{1}{2}\int_{\partial D}(x)dy - (y)dx$

$$= \frac{1}{2}\int_0^{2\pi}[(acost)(bcost) - (bsint)(-asint)]\, dt$$

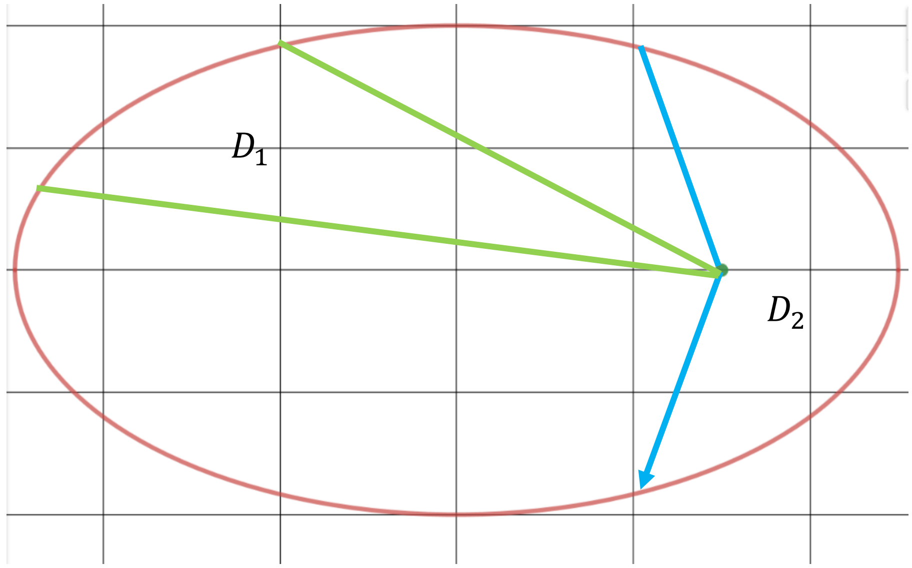
$$= \frac{1}{2}\int_0^{2\pi} ab(\cos^2 t + \sin^2 t)dt$$

$$= \frac{1}{2}\int_0^{2\pi}(ab)dt = \pi ab.$$

Ex. Green's Theorem and Kepler's 2nd law of planetary motion.

Kepler's 2nd law of planetary motion says that a line segment joining a planet and the sun sweeps out equal areas during equal intervals of time.

Area(D_1) =Area(D_2).



By our previous theorem we know that :

$A = \frac{1}{2}\int_{\partial D}(x)dy - (y)dx$; where c=∂D is given by $\vec{c}(t) =< x(t), y(t) >$.

So we can rewrite the area A by using: $dx = x'(t)dt$, $dy = y'(t)dt$.

$$A = \frac{1}{2}\int_{t_0}^{t_1}(x(t))(y'(t))dt - (y(t))(x'(t))dt$$

$$A = \frac{1}{2}\int_{t_0}^{t_1}[(x(t))(y'(t)) - (y(t))(x'(t))]dt .$$

So Kepler's 2nd law of planetary motion folows if we can show that the integrand,

$(x(t))(y'(t)) - (y(t))(x'(t))$, is a constant.

This will follow from Newton's law of gravity ($\vec{F} = -MmG < \frac{x}{(x^2+y^2)^{\frac{3}{2}}}, \frac{y}{(x^2+y^2)^{\frac{3}{2}}} >$)

and Newton's second law of motion ($\vec{F} = m\vec{a} = m < x''(t), y''(t) >$).

Setting the 2 expressions for $\vec{F}$ equal to eachother we get:

$$x''(t) = -MG\left(\frac{x}{(x^2+y^2)^{\frac{3}{2}}}\right) \qquad y''(t) = -MG\left(\frac{y}{(x^2+y^2)^{\frac{3}{2}}}\right).$$

Now let's show that $\big(x(t)\big)\big(y'(t)\big) - \big(y(t)\big)\big(x'(t)\big)$ is a constant:

$$\frac{d}{dt}\Big(\big(x(t)\big)\big(y'(t)\big) - \big(y(t)\big)\big(x'(t)\big)\Big) = x(t)y''(t) - y(t)x''(t)$$

$$= x(t)\left(-MG\left(\frac{y}{(x^2+y^2)^{\frac{3}{2}}}\right)\right) - y(t)\left(-MG\left(\frac{x}{(x^2+y^2)^{\frac{3}{2}}}\right)\right) = 0.$$

Vector Form of Green's Theorem: Let $D \subset \mathbb{R}^2$ be a region to which Green's theorem applies. Let ∂D be its positively oriented (ie counterclockwise) boundary, and let $\vec{F} = P(x,y)\vec{\imath} + Q(x,y)\vec{\jmath}$ be a C^1 vector field on D then

$$\int_{\partial D} \vec{F} \cdot d\vec{s} = \iint_D (curl\,\vec{F}) \cdot \vec{k}\,dA = \iint_D (\nabla \times \vec{F}) \cdot \vec{k}\,dA.$$

Proof: $\vec{F} = P(x,y)\vec{\imath} + Q(x,y)\vec{\jmath} + 0\vec{k}$.

$$curl\vec{F} = \nabla \times \vec{F} = \begin{vmatrix} \vec{\imath} & \vec{\jmath} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & 0 \end{vmatrix} = -\frac{\partial Q}{\partial z}\vec{\imath} - \frac{\partial P}{\partial z}\vec{\jmath} + \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\vec{k}$$

Thus $\big(\nabla \times \vec{F}\big) \cdot \vec{k} = \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}$.

$$\vec{F} \cdot d\vec{s} = (P\vec{\imath} + Q\vec{\jmath}) \cdot (dx\vec{\imath} + dy\vec{\jmath}) = Pdx + Qdy.$$

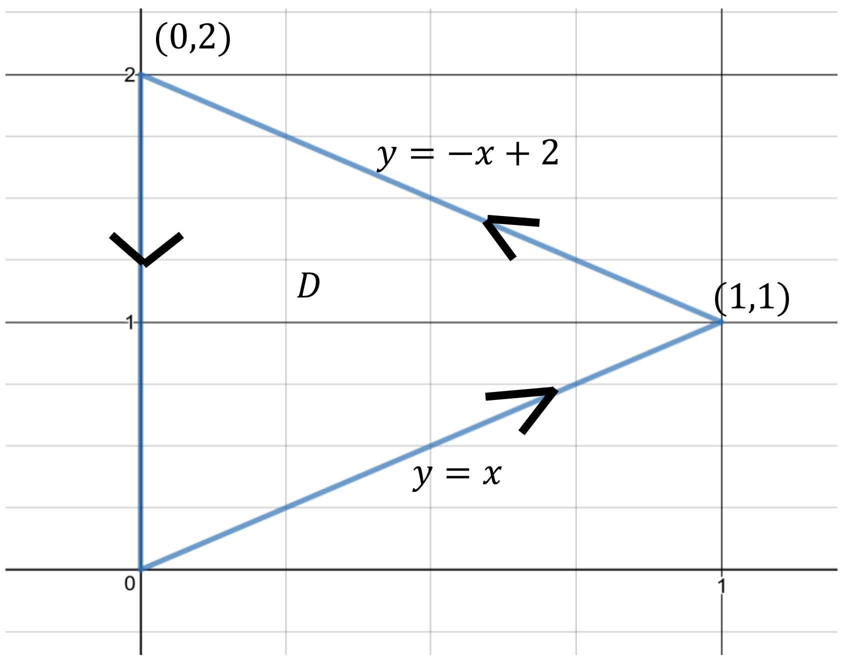
So by Green's theorem:

$$\int_C P(x,y)dx + Q(x,y)dy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)dxdy$$ or

$$\int_{\partial D} \vec{F} \cdot d\vec{s} = \iint_D (curl\,\vec{F}) \cdot \vec{k}dA = \iint_D (\nabla \times \vec{F}) \cdot \vec{k}dA\,.$$

Ex. Find the work done to move a particle around a triangle (counterclockwise) with vertices $(0,0)$, $(1,1)$, and $(0,2)$, if the force vector field is given by

$$\vec{F}(x,y) = (cosx + xy)\vec{\imath} + (x^2 + e^{siny})\vec{\jmath}\,.$$



In this case $P(x,y) = cosx + xy,\quad Q(x,y) = x^2 + e^{siny}$,

So $\dfrac{\partial P}{\partial y} = x,\quad \dfrac{\partial Q}{\partial x} = 2x.$

So by Green's theorem:

Work$= \int_{\partial D} \vec{F} \cdot d\vec{s} = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dxdy$

$= \iint_D (2x - x)dxdy = \iint_D x\, dxdy$

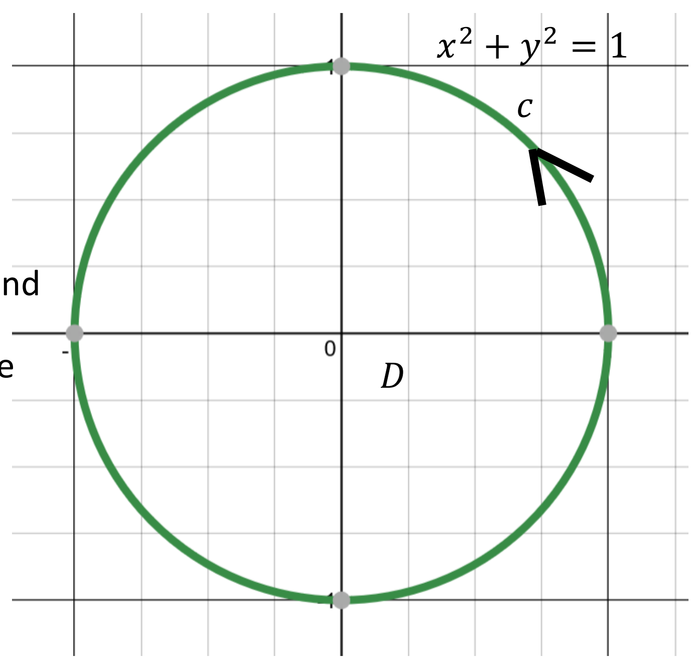
$= \int_{x=0}^{x=1} \int_{y=x}^{y=-x+2} x\, dydx$

$= \int_{x=0}^{x=1} xy\Big|_{y=x}^{y=-x+2} dx$

$= \int_{x=0}^{x=1} [x(-x+2) - x^2]dx$

$= \int_{x=0}^{x=1} (2x - 2x^2)dx = (x^2 - \frac{2}{3}x^3)\Big|_0^1 = 1 - \frac{2}{3} = \frac{1}{3}.$

Ex. Let $\vec{F}(x,y) = (2y + e^x)\vec{\imath} + (x + \sin(y^2))\vec{\jmath}$, and let c be the circle $x^2 + y^2 = 1$. Evaluate $\int_c \vec{F} \cdot d\vec{s}$.



In this case, $P(x,y) = 2y + e^x$ and $Q(x,y) = x + \sin(y^2)$, so we have $\frac{\partial P}{\partial y} = 2$ and $\frac{\partial Q}{\partial x} = 1$.

So by Green's theorem:

$$\int_c \vec{F} \cdot d\vec{s} = \iint_D (\nabla \times \vec{F}) \cdot \vec{k} dA = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dxdy = \iint_D (1-2) dxdy$$

$= \iint_D -1\, dxdy;$ Where D is the unit disk about the origin.

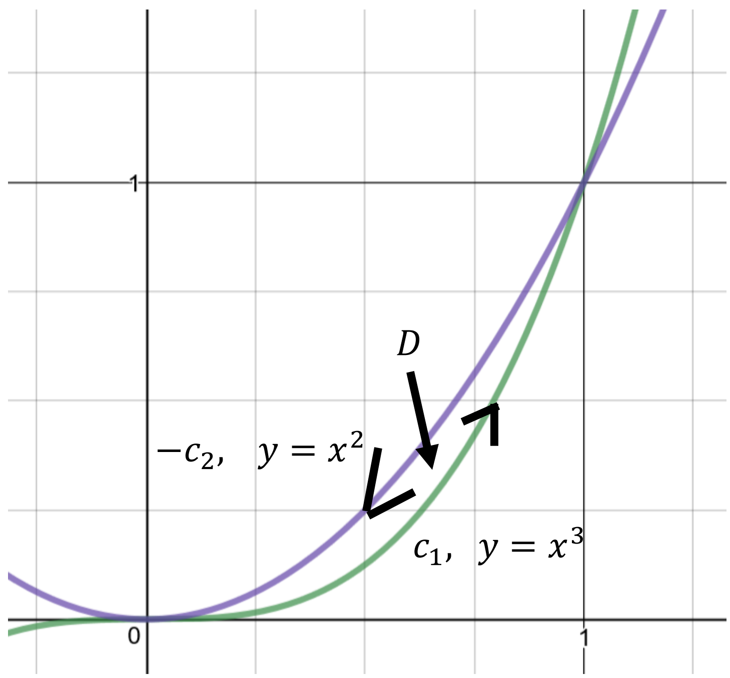
$$= -(area\ of\ unit\ disk) = -\pi.$$

Ex. Evaluate $\iint_D (\nabla \times \vec{F}) \cdot \vec{k} dA$ using Green's theorem where

$\vec{F}(x, y) =< x^2 + y^2, xy >$, and D is the region in the first quadrant bounded by $y = x^3$ and $y = x^2$.

First draw the region D:

$\partial D = c_1 - c_2$



$$\iint_D (\nabla \times \vec{F}) \cdot \vec{k} dA = \int_{\partial D} \vec{F} \cdot d\vec{s} = \int_{\partial D} < x^2 + y^2, xy > \cdot < dx, dy >$$

$$= \int_{c_1} (x^2 + y^2)\, dx + (xy) dy - \int_{c_2} (x^2 + y^2)\, dx + (xy) dy.$$

Now parametrize the curves c_1 and c_2 and compute the line integrals.

$\vec{c}_1(t) =< t,\ t^3 >; \qquad 0 \le t \le 1$

$\vec{c}_1'(t) =< 1, 3t^2 >$

$\vec{c}_2(t) =< t,\ t^2 >; \qquad 0 \le t \le 1$

$\vec{c}_2'(t) =< 1, 2t >$

$$\iint_D (\nabla \times \vec{F}) \cdot \vec{k}\, dA = \int_0^1 [(t^2 + t^6) + t^4(3t^2)]dt - \int_0^1 [(t^2 + t^4) + t^3(2t)]dt$$

$$= \int_0^1 (t^2 + 4t^6)dt - \int_0^1 (t^2 + 3t^4)dt$$

$$= \int_0^1 (4t^6 - 3t^4)dt = (\tfrac{4}{7}t^7 - \tfrac{3}{5}t^5)\Big|_0^1 = \tfrac{4}{7} - \tfrac{3}{5} = -\tfrac{1}{35}.$$

Divergence form of Green's theorem (This can be generalized to $\mathbb{R}^3$).

Let $D \subset \mathbb{R}^2$ be a region to which Green's theorem applies. Let ∂D be its positively oriented (ie counterclockwise) boundary. Let $\vec{n}$ denote the outward unit normal to ∂D. Let $\vec{F} = R(x,y)\vec{i} + T(x,y)\vec{j}$ be a C^1 vector field on D then:

$$\int_{\partial D} (\vec{F} \cdot \vec{n}) ds = \iint_D (div\, \vec{F}) dA.$$

Proof: For any vector $\vec{v} =< \alpha, \beta >$, in $\mathbb{R}^2$, both $< \beta, -\alpha >$ and $< -\beta, \alpha >$ are perpendicular to $\vec{v}$.

So If $\partial D = \vec{c}(t) =< x(t), y(t) >$, then $\vec{c}\,'(t) =< x'(t), y'(t) >$ is a tangent vector to $\vec{c}(t)$. An outward pointing normal vector is given by $\vec{N}(t) =< y'(t), -x'(t) >$.

Thus the outward pointing unit normal vector is given by: $\vec{n}(t) = \dfrac{< y'(t), -x'(t) >}{\sqrt{(x'(t))^2 + (y'(t))^2}}$

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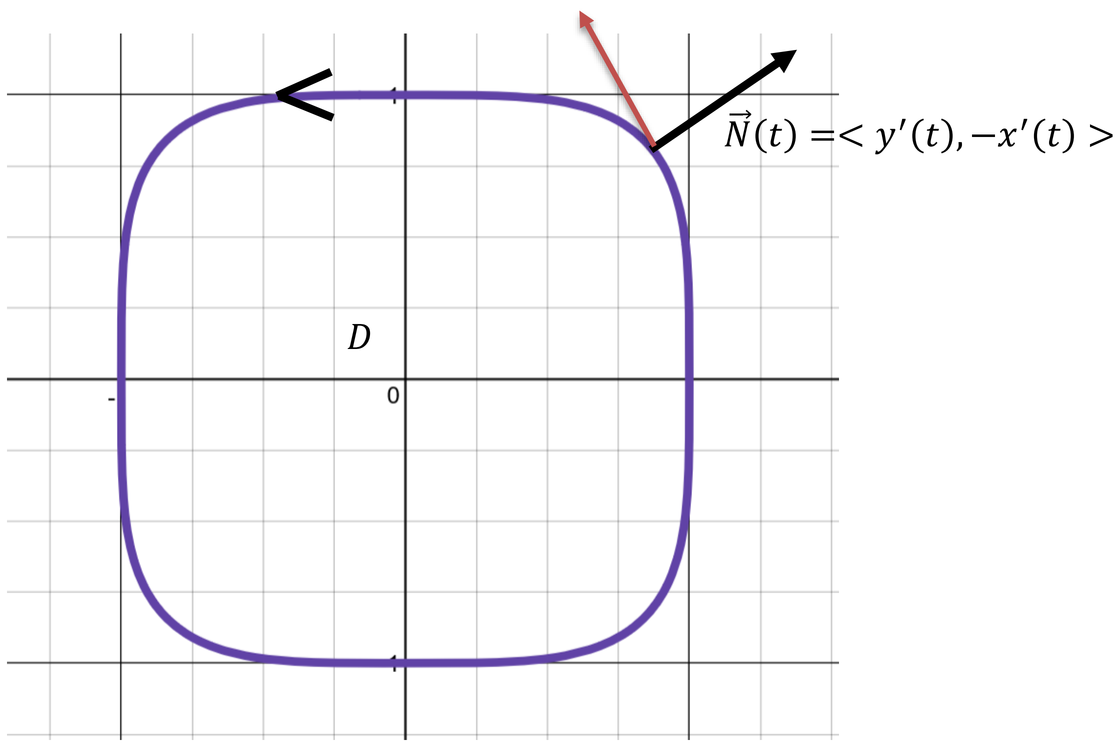


Since $|\vec{c}'(t)| = \sqrt{(x'(t))^2 + (y'(t))^2}$ we have:

$\int_{\partial D}(\vec{F} \cdot \vec{n})ds =$

$$= \int_a^b < R(x(t), y(t)), T(x(t), y(t)) > \cdot \frac{<y'(t), -x'(t)>}{\sqrt{(x'(t))^2 + (y'(t))^2}} \sqrt{(x'(t))^2 + (y'(t))^2} dt$$

$$= \int_a^b [R(x(t), y(t))y'(t) - T(x(t), y(t))x'(t)]dt .$$

Since $dx = x'(t)dt, \;\; dy = y'(t)dt$ we have

$$= \int_{\partial D} R(x, y)dy - T(x, y)dx.$$

Now applying the standard Green's theorem where $P(x,y) = -T(x,y)$, $Q(x,y) = R(x,y)$ we get

$$\int_{\partial D}(\vec{F}\cdot\vec{n})ds = \iint_D \left(\frac{\partial R}{\partial x} + \frac{\partial T}{\partial y}\right) dxdy = \iint_D (div\vec{F})dxdy\ .$$

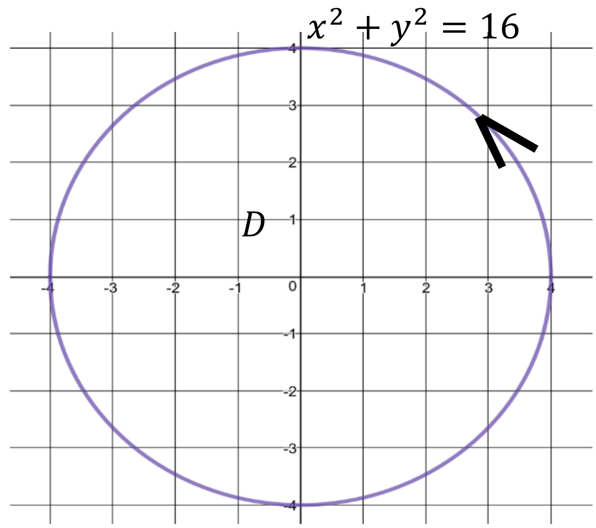
Ex. Let $\vec{F}(x,y) = (siny)\vec{\imath} + (cosx)\vec{\jmath}$. Find $\int_{\partial D}(\vec{F}\cdot\vec{n})ds$, where ∂D is the unit circle.

$$\int_{\partial D}(\vec{F}\cdot\vec{n})ds = \iint_D (div\,\vec{F})dA$$

But $div\vec{F} = 0$, so $\iint_D (div\,\vec{F})dA = 0$.

Thus $\int_{\partial D}(\vec{F}\cdot\vec{n})ds = 0$.

Ex. Verify the Divergence thm for $\vec{F} = (xy)\vec{\imath} - (y)\vec{\jmath}$ and the disk $x^2 + y^2 \leq 16$.



First let's calculate $\int_{\partial D}(\vec{F}\cdot\vec{n})ds$.

$\partial D = \vec{c}(t) =< 4cost, 4sint >;\quad 0 \leq t \leq 2\pi$

$\vec{c}'(t) =< -4sint, 4cost >$

$|\vec{c}'(t)| = \sqrt{16\cos^2 t + 16\sin^2 t} = 4$

$\vec{n}(t) = \frac{<4cost,\ 4sint>}{4} =< cost, sint >$

$\vec{F}\cdot\vec{n} =< 16costsint, -4sint >\cdot< cost, sint > = 16(\cos^2 t)(sint) - 4\sin^2 t$

$\int_{\partial D}(\vec{F}\cdot\vec{n})ds = \int_0^{2\pi}[16\,(\cos^2 t)(sint) - 4\sin^2 t](4)dt$

$$= 16\left[\int_0^{2\pi} 4\,(\cos^2 t)(sint)dt - \int_0^{2\pi}(\sin^2 t)\,dt\right]$$

If we let $u = cost$ in the first integral we see it's equal to 0.

Substituting $\sin^2 t = \frac{1}{2} - \frac{1}{2}cos2t$ in the second integral shows it equals π.

$\int_{\partial D}(\vec{F}\cdot\vec{n})ds = -16\pi.$

Now let's calculate $\iint_D\,(div\vec{F})dxdy$.

$div\vec{F} = y - 1\,.$

$\iint_D\,(div\vec{F})dxdy = \iint_D\,(y-1)dxdy.$

Now since D is a disk, change to polar coordinates.

$\iint_D\,(div\vec{F})dxdy = \int_0^{2\pi}\int_0^4 (rsin\theta - 1)\,(r)drd\theta$

$$= \int_0^{2\pi}\int_0^4 (r^2 sin\theta - r)\,drd\theta$$

$$= \int_0^{2\pi}(\tfrac{1}{3}r^3 sin\theta - \tfrac{1}{2}r^2)\Big|_0^4\,d\theta$$

$$= \int_0^{2\pi}(\tfrac{64}{3}sin\theta - 8)d\theta$$

$$= \frac{-64}{3}cos\theta - 8\theta\Big|_0^{2\pi} = -16\pi.$$