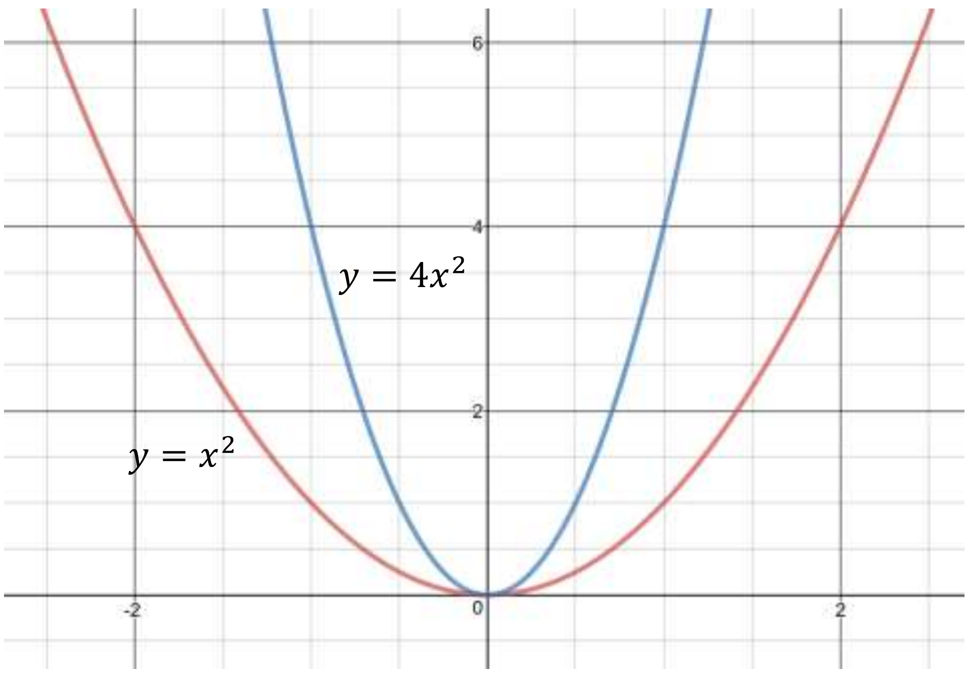
Gauss Curvature and the Gauss-Bonnet Theorem

Let $\vec{\Phi}: D \to \mathbb{R}^3$, be a smooth parametrized surface S ($\vec{\Phi}(D) = S,\ \ \vec{T}_u \times \vec{T}_v \neq 0$).

We want to develop a definition of the Gauss Curvature of a surface at a point $p \in S$.

Let's start with 2 curves in the x, y plane: $y = x^2$ and $y = 4x^2$.



Our intuition tells us that the curvature of $y = 4x^2$ at the point $(0,0)$ is larger than the curvature of $y = x^2$ at $(0,0)$. We can also intuitively conclude that the notion of curvature should be related to the second derivative of the function at that point. Not surprisingly, $y'' = 8$ for the curve $y = 4x^2\ at\ (0,0)$, and $y'' = 2$ for the curve $y = x^2$ at $(0,0)$. (In this particular example, the curvature of the 2 curves actually does equal 8 and 2, however, in general, the calculation of curvature is a more complicated than just calculating the 2nd derivative.)

Now let's parametrize both curves:

$\overrightarrow{c_1}(t) =< t, t^2 >$ $\overrightarrow{c_2}(t) =< t, 4t^2 >$

$\vec{c}_1{}'(t) =< 1,2t >$ $\vec{c}_2{}'(t) =< 1, 8t >$

$\vec{c}_1{}''(t) =< 0,2 >$ $\vec{c}_2{}''(t) =< 0,8 >$

Notice the upward pointing unit normal vector at the point (0,0) to both curves is

$\vec{n} =< 0,1 >$.

Finally, notice that the "curvatures" we calculated above can be gotten by:

$\vec{c}_1{}''(t) \cdot \vec{n} =< 0,2 > \cdot < 0,1 > = 2$

$\vec{c}_2{}''(t) \cdot \vec{n} =< 0,8 > \cdot < 0,1 > = 8.$

Thus it should not surprise us to see 2nd derivatives dotted with a unit normal vector in a formula for curvature of a surface.

To define curvature for a surface we start by defining:

$$E = \left|\frac{\partial\vec{\Phi}}{\partial u}\right|^2, \quad F = \frac{\partial\vec{\Phi}}{\partial u} \cdot \frac{\partial\vec{\Phi}}{\partial v}, \qquad G = \left|\frac{\partial\vec{\Phi}}{\partial v}\right|^2$$

It can be shown through a messy calculation that:

$$\left|\vec{T}_u \times \vec{T}_v\right|^2 = EG - F^2.$$

Let $W = EG - F^2$.

We know that a unit normal vector, $\vec{n}$, on the surface S is given by:

$$\vec{n} = \frac{\vec{T}_u \times \vec{T}_v}{\left|\vec{T}_u \times \vec{T}_v\right|} = \frac{\vec{T}_u \times \vec{T}_v}{\sqrt{W}}.$$

Now we define 3 functions l, m, n at a point $p \in S$ by:

$$l(p) = \vec{n}(u,v) \cdot \frac{\partial^2 \vec{\Phi}}{\partial u^2}$$

$$m(p) = \vec{n}(u,v) \cdot \frac{\partial^2 \vec{\Phi}}{\partial u \partial v}$$

$$n(p) = \vec{n}(u,v) \cdot \frac{\partial^2 \vec{\Phi}}{\partial v^2}$$

We now define the **Gauss Curvature**, $k(p)$, for $p \in S$:

$$\boldsymbol{k(p) = \frac{ln - m^2}{W}}\ .$$

Ex. Planes have curvature 0 at every point.

The general form of a plane in $\mathbb{R}^3$ has the form:

$$\vec{\Phi}(u,v) = < (a_1 u + a_2 v + a_3),\ (b_1 u + b_2 v + b_3),\ (c_1 u + c_2 v + c_3) > ;$$

where a_i, b_i, c_i are constants.

Notice that since $\vec{\Phi}(u,v)$ is linear in u and v we have:

$$\frac{\partial^2 \vec{\Phi}}{\partial u^2} = 0, \qquad \frac{\partial^2 \vec{\Phi}}{\partial u \partial v} = 0, \qquad \frac{\partial^2 \vec{\Phi}}{\partial v^2} = 0 \ \Longrightarrow$$

$$l(p) = \vec{n}(u,v) \cdot \frac{\partial^2 \vec{\Phi}}{\partial u^2} = 0$$

$$m(p) = \vec{n}(u,v) \cdot \frac{\partial^2 \vec{\Phi}}{\partial u \partial v} = 0$$

$$n(p) = \vec{n}(u,v) \cdot \frac{\partial^2 \vec{\Phi}}{\partial v^2} = 0.$$

Thus we have:

$$k(p) = \frac{ln - m^2}{W} = 0.$$

Ex. Calculate the Gauss curvature of the sphere $x^2 + y^2 + z^2 = R^2$.

$\vec{\Phi}(\phi, \theta) =< R\ cos\theta sin\phi,\ Rsin\theta sin\phi,\ Rcos\phi >;\ 0 \le \phi \le \pi,$ and $0 \le \theta \le 2\pi.$

$\vec{T}_\phi =< Rcos\theta cos\phi, Rsin\theta cos\phi, -Rsin\phi >$

$\vec{T}_\theta =< -Rsin\theta sin\phi, Rcos\theta sin\phi, 0 >$

$$\vec{T}_\phi \times \vec{T}_\theta = \begin{vmatrix} \vec{\imath} & \vec{\jmath} & \vec{k} \\ Rcos\theta cos\phi & Rsin\theta cos\phi & -Rsin\phi \\ -Rsin\theta sin\phi & Rcos\theta sin\phi & 0 \end{vmatrix}$$

$$\vec{T}_\phi \times \vec{T}_\theta = R^2 sin^2\phi cos\theta \vec{\imath} + R^2 sin^2\phi sin\theta \vec{\jmath} + R^2 sin\phi cos\phi \vec{k}.$$

$$\left|\vec{T}_\phi \times \vec{T}_\theta\right| = \sqrt{R^4 sin^4\phi cos^2\theta + R^4 sin^4\phi sin^2\theta + R^4 sin^2\phi cos^2\phi}$$

$$= R^2 sin\phi.$$

Unit normal=$\vec{n} = \frac{\vec{T}_\phi \times \vec{T}_\theta}{\left|\vec{T}_\phi \times \vec{T}_\theta\right|};$

$$= \frac{R^2 sin^2\phi cos\theta \vec{\imath} + R^2 sin^2\phi sin\theta \vec{\jmath} + R^2 sin\phi cos\phi \vec{k}}{R^2 sin\phi}$$

$$\vec{n} = sin\phi cos\theta \vec{\imath} + sin\phi sin\theta \vec{\jmath} + cos\phi \vec{k}\ .$$

$$\frac{\partial^2\vec{\Phi}}{\partial\phi^2} =< -Rcos\theta sin\phi, -Rsin\theta sin\phi, -Rcos\phi >$$

$$\frac{\partial^2\vec{\Phi}}{\partial\phi\partial\theta} =< -Rsin\theta cos\phi, Rcos\theta cos\phi,\ 0 >$$

$$\frac{\partial^2\vec{\Phi}}{\partial\theta^2} =< -Rcos\theta sin\phi, -Rsin\theta sin\phi, 0 >$$

$$l(p) = \vec{n} \cdot \frac{\partial^2\vec{\Phi}}{\partial\phi^2}$$

$$=< sin\phi cos\theta, sin\phi sin\theta, cos\phi > \cdot < -Rcos\theta sin\phi, -Rsin\theta sin\phi, -Rcos\phi >$$

$$= -Rcos^2\theta \sin^2\phi - Rsin^2\theta \sin^2\phi - Rcos^2\phi$$

$$= -R[\sin^2\phi(\cos^2\theta + \sin^2\theta) + \cos^2\phi] = -R$$

$$m(p) = \vec{n} \cdot \frac{\partial^2\vec{\Phi}}{\partial\phi\partial\theta}$$

$$=< sin\phi cos\theta, sin\phi sin\theta, cos\phi > \cdot < -Rsin\theta cos\phi, Rcos\theta cos\phi,\ 0 >$$

$$= 0$$

$$n(p) = \vec{n} \cdot \frac{\partial^2\vec{\Phi}}{\partial\theta^2}$$

$$=< sin\phi cos\theta, sin\phi sin\theta, cos\phi > \cdot < -Rcos\theta sin\phi, -Rsin\theta sin\phi, 0 >$$

$$= -Rsin^2\phi \cos^2\theta - Rsin^2\phi \sin^2\theta$$

$$= -Rsin^2\phi(\cos^2\theta + \sin^2\theta) = -Rsin^2\phi.$$

Since $\left|\vec{T}_\phi \times \vec{T}_\theta\right| = R^2 sin\phi$,

$W = \left|\vec{T}_\phi \times \vec{T}_\theta\right|^2 = R^4 sin^2\phi$.

Finally, we have the Gauss Curvature, $k(p)$:

$$\boldsymbol{k(p) = \frac{ln - m^2}{W} = \frac{(-R)(-Rsin^2\phi) - (0)^2}{R^4 sin^2\phi} = \frac{1}{R^2}}$$

Thus the Gauss Curvature of a sphere of radius R is constant at all points and equal to $\frac{1}{R^2}$.

Notice that for a sphere of any radius R:

$$\frac{1}{2\pi}\iint_S \; k(p)dS = \frac{1}{2\pi}\iint_S \; \frac{1}{R^2}dS = \frac{1}{2\pi R^2}\iint_S \; dS = \frac{4\pi R^2}{2\pi R^2} = 2.$$

$\iint_S \; k(p)dS$ is sometimes referred to as the total curvature of the surface S.

Gauss-Bonnet Theorem: $\frac{1}{2\pi}\iint_S \; k(p)dS = 2 - 2g$, where S is a closed smooth surface in $\mathbb{R}^3$ and g is the genus (number of holes) of the surface.

Thus, $\frac{1}{2\pi}\iint_S \; k(p)dS$ for any ellipsoid is equal to $\frac{1}{2\pi}\iint_S \; k(p)dS$ for any sphere $(= 2)$, since both surfaces are genus 0 (no holes). In fact, this remarkable theorem says that if we had a sphere made of clay and distorted it into any smooth surface we like without punching a hole in it, $\frac{1}{2\pi}$ times the integral of the Gauss curvature of that new surface over that surface (ie $\frac{1}{2\pi}$ times the total curvature) would still be 2.