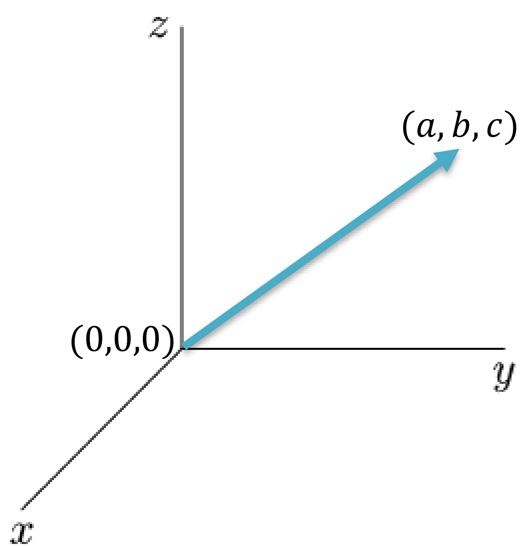
A Quick Review of a few Topics from 3rd Semester Calculus

1. Vectors in R^3

A vector in R^3 is a line segment from the origin $(0,0,0)$ to a point in R^3, (a, b, c). We denote this vector by $< a, b, c >$.



We can also write this vector as:

$< a, b, c >= a\vec{\imath} + b\vec{\jmath} + c\vec{k};$

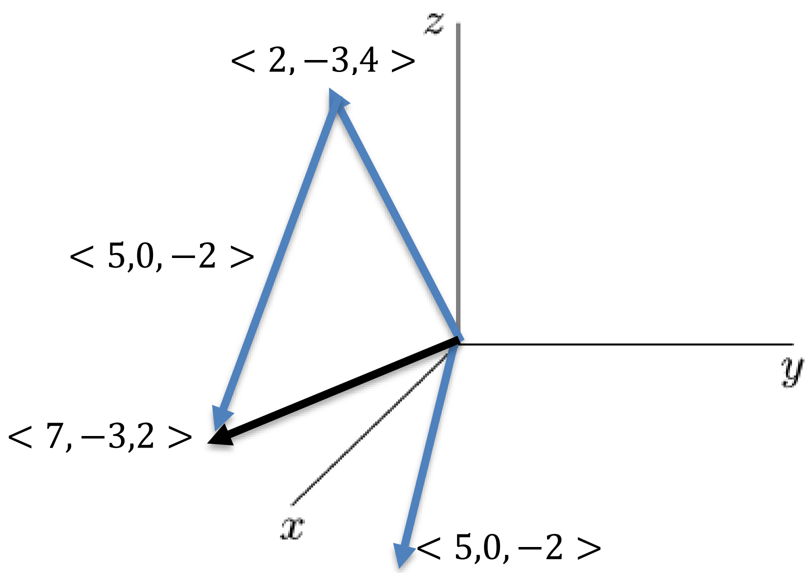
where $\vec{\imath} =< 1,0,0 >$, $\vec{\jmath} =< 0,1,0 >$, $\vec{k} =< 0,0,1 >$.

Ex. $< 2,5,-1 >= 2\vec{\imath} + 5\vec{\jmath} - \vec{k}$.

We can add or subtract vectors by adding or subtracting their components.

Ex. $< 2, -3, 4 > +< 5, 0, -2 >=< 7, -3, 2 >$

$< 3, 2, -4 > -< 5, -1, 2 > =< -2, 3, -6 >$



We can also multiply a vector by a real number (called a scalar), by multiplying each of the components.

Ex. $(-6) < 3, -2, -3 > = < -18, 12, 18 >$.

There are 2 ways to multiply vectors in R^3, through a "Dot" product (whose answer is a number, not a vector), and through a "Cross" product (whose answer is a vector not a number).

Let $\vec{v}_1 = < a_1, b_1, c_1 >$ and $\vec{v}_2 = < a_2, b_2, c_2 >$.

Dot Product:

$\vec{v}_1 \cdot \vec{v}_2 = a_1 a_2 + b_1 b_2 + c_1 c_2$

Note: $\vec{v}_1 \cdot \vec{v}_2$ is a real number, NOT a vector.

Ex. $< 2, -3,4 > \cdot < 5,0, -2 > = (2)(5) + (-3)(0) + (4)(-2) = 10 + 0 - 8 = 2$.

Notice that: $\vec{v}_1 \cdot \vec{v}_1 = {a_1}^2 + {b_1}^2 + {c_1}^2 = \|\vec{v}_1\|^2$

or $\|\vec{v}_1\| = \sqrt{\vec{v}_1 \cdot \vec{v}_1} = \sqrt{{a_1}^2 + {b_1}^2 + {c_1}^2}$

Properties of the Dot product:

1. $\vec{v}_1 \cdot \vec{v}_2 = \vec{v}_2 \cdot \vec{v}_1$
2. $\vec{v}_1 \cdot (\vec{v}_2 + \vec{v}_3) = \vec{v}_1 \cdot \vec{v}_2 + \vec{v}_1 \cdot \vec{v}_3$

If $\vec{v} = a\vec{\imath} + b\vec{\jmath} + c\vec{k}$, $\vec{v} \neq \vec{0}$, then a unit vector (a vector of length 1) in the direction of $\vec{v}$ is given by:

$$\vec{u} = \frac{\vec{v}}{\|\vec{v}\|} = \frac{a}{\sqrt{a^2+b^2+c^2}}\vec{\imath} + \frac{b}{\sqrt{a^2+b^2+c^2}}\vec{\jmath} + \frac{c}{\sqrt{a^2+b^2+c^2}}\vec{k}$$

Ex. Find a unit vector in the direction of $\vec{v} = < 2, -2, 1 > = 2\vec{\imath} - 2\vec{\jmath} + \vec{k}$

Here $a = 2$, $b = -2$, $c = 1$, so $a^2 + b^2 + c^2 = 4 + 4 + 1 = 9$.

$$\vec{u} = \frac{a}{\sqrt{a^2+b^2+c^2}}\vec{\imath} + \frac{b}{\sqrt{a^2+b^2+c^2}}\vec{\jmath} + \frac{c}{\sqrt{a^2+b^2+c^2}}\vec{k} = \frac{2}{3}\vec{\imath} - \frac{2}{3}\vec{\jmath} + \frac{1}{3}\vec{k}$$

Theorem: Assume $\vec{v}, \vec{w} \neq \vec{0}$. Then $\vec{v} \cdot \vec{w} = 0$ if and only if $\vec{v}$ and $\vec{w}$ are perpendicular.

Cross Product:

$\vec{v}_1 =< a_1, b_1, c_1 > = a_1\vec{i} + b_1\vec{j} + c_1\vec{k}$

$\vec{v}_2 =< a_2, b_2, c_2 > = a_2\vec{i} + b_2\vec{j} + c_2\vec{k}$

$$\vec{v}_1 \times \vec{v}_2 = det\begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix}$$ (Note: The "det" is often omitted)

Ex. Find $< 2,1,-3 > \times < -1,1,2 >$

$$< 2,1,-3 > \times < -1,1,2 > = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 2 & 1 & -3 \\ -1 & 1 & 2 \end{vmatrix}$$

$$= \begin{vmatrix} 1 & -3 \\ 1 & 2 \end{vmatrix}\vec{i} - \begin{vmatrix} 2 & -3 \\ -1 & 2 \end{vmatrix}\vec{j} + \begin{vmatrix} 2 & 1 \\ -1 & 1 \end{vmatrix}\vec{k}$$

$$= [(1)(2) - (1)(-3)]\vec{i} - [(2)(2) - (-1)(-3)]\vec{j} + [(2)(1) - (-1)(1)]\vec{k}$$

$$= 5\vec{i} - \vec{j} + 3\vec{k}.$$

Notice that the answer is a vector, NOT a number (ie a scalar).

Properties:

1. $\vec{v} \times \vec{w}$ is perpendicular to $\vec{v}$ and $\vec{w}$ (hence perpendicular to the plane containing $\vec{v}$ and $\vec{w}$)

2. $\vec{v} \times \vec{w} = 0$ if and only if $\vec{v}$ and $\vec{w}$ are parallel or $\vec{v} = 0$ or $\vec{w} = 0$.

3. $\vec{v} \times \vec{w} = -\vec{w} \times \vec{v}$.

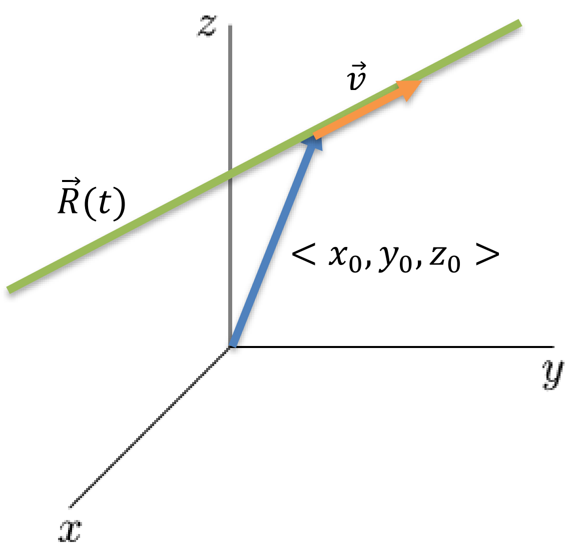
2. Finding an equation of a line in R^3

Given a point (x_0, y_0, z_0) and a direction vector $\vec{v} =< a, b, c >$, we can write a vector equation of a line through (x_0, y_0, z_0) in the direction of $\vec{v} =< a, b, c >$ by:

$\vec{R}(t) =< x_0, y_0, z_0 > +t\vec{v} =< x_0, y_0, z_0 > +t < a, b, c >; \quad t \in \mathbb{R}$.

This **vector form of a line** can also be written as:

$\vec{R}(t) =< (x_0 + at), (y_0 + bt), (z_0 + ct) >; \quad t \in \mathbb{R}$.



This line can also be written in parametric form:

$x = x_0 + at$

$y = y_0 + bt$

$z = z_0 + ct$

where $t \in \mathbb{R}$.

Ex. Find a vector equation and parametric equations for the line through the points $P = (2, -3, -1)$ and $Q = (-1,2,3)$.

First we find a direction vector from P to Q (or from Q to P)

Direction Vector $\vec{v} = \overrightarrow{PQ} = \vec{Q} - \vec{P} =< -1 - 2, 2 - (-3), 3 - (-1) >$

$$=< -3,5,4 >.$$

Now we use either point, say, $(2, -3, -1) = (x_0, y_0, z_0)$:

$\vec{R}(t) =< x_0 + at, y_0 + bt,\ z_0 + ct > =< 2 - 3t,\ -3 + 5t,\ -1 + 4t >;\ \ t \in \mathbb{R}$.

In parametric equations this becomes:

$x = x_0 + at = 2 - 3t$

$y = y_0 + at = -3 + 5t$

$z = z_0 + at = -1 + 4t$

where $t \in \mathbb{R}$.

Equations of line segments from P to Q.

If we find an equation of the line through the points P and Q by finding the direction vector $\vec{v} = \overrightarrow{PQ} =< a, b, c >= \vec{Q} - \vec{P}$, and use the starting point $P = (x_0, y_0, z_0)$ as our point on the line then the vector equation of the line is

$$\vec{R}(t) =< x_0 + at, y_0 + bt,\ z_0 + ct >;\ \ t \in \mathbb{R}.$$

If the equation of the line through P and Q is found this way (there are an infinite number of equations of lines that go through P and Q) then the line segment from P to Q is given by

$$\vec{R}(t) = < x_0 + at,\ y_0 + bt,\ z_0 + ct >; \quad 0 \le t \le 1.$$

Notice that at $t = 0$; $\vec{R}(0) = < x_0, y_0, z_0 > = \vec{P}$

$$t = 1; \quad \vec{R}(1) = < x_0 + a, y_0 + b, z_0 + c > = \vec{Q}$$

since $\vec{v} = \overrightarrow{PQ} = \vec{Q} - \vec{P}$.

Alternatively, we can find a line segment from P to Q by taking:

$\vec{R}(t) = t\vec{P} + (1-t)\vec{Q}$; $0 \le t \le 1$ (Starts at Q, end at P); or

$\vec{R}(t) = t\vec{Q} + (1-t)\vec{P}$; $0 \le t \le 1$ (Starts at P, end at Q).

Ex. Find an equation for the line segment between $P = (2, -3, -1)$ and $Q = (-1,2,3)$.

In the previous example we found an equation of the line through P and Q by finding the direction $\vec{v} = \overrightarrow{PQ} = < a, b, c > = \vec{Q} - \vec{P}$, and using the starting point $P = (2, -3, -1)$. Thus an equation for the line segment between P and Q is

$$\vec{R}(t) = < x_0 + at, y_0 + bt,\ z_0 + ct >$$

$$= < 2 - 3t, -3 + 5t, -1 + 4t >; \quad 0 \le t \le 1.$$

Using the second approach we could find an equation for the line segment by

$$\vec{R}(t) = t\vec{P} + (1-t)\vec{Q} = t < 2, -3, -1 > +(1-t) < -1,2,3 >; \quad 0 \le t \le 1$$

$$= < -1 + 3t,\ 2 - 5t,\ 3 - 4t >; \qquad 0 \le t \le 1 \text{ (Starts at } Q\text{)}$$

or

$$\vec{R}(t) = t\vec{Q} + (1-t)\vec{P} = t < -1,2,3 > +(1-t) < 2, -3, -1 >; \quad 0 \le t \le 1$$

$$= < 2 - 3t,\ -3 + 5t,\ -1 + 4t >; \ 0 \le t \le 1. \quad \text{(Starts at } P\text{).}$$

3. Equations of planes in R^3

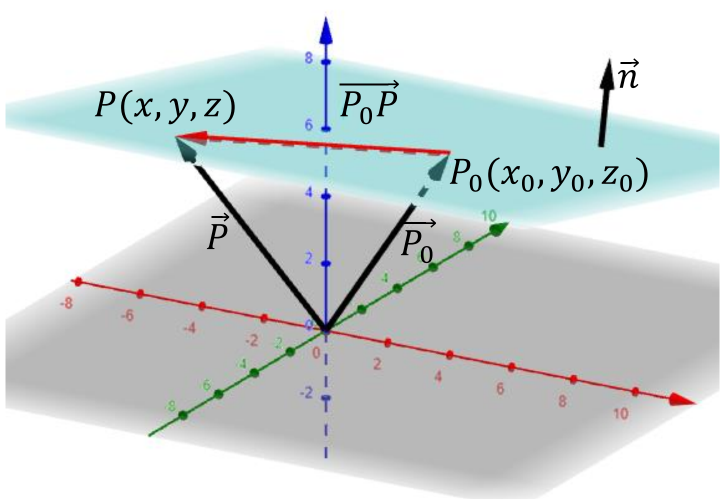
In order to write an equation for a plane in R^3 we need a point (x_0, y_0, z_0) and a vector, $\vec{n}$, perpendicular to the plane, called a "normal" vector.

For any general point (x, y, z) on the plane we have:

$\vec{P} = < x, y, z >, \quad \vec{P_0} = < x_0, y_0, z_0 >, \quad \vec{n} = < A, B, C >,$

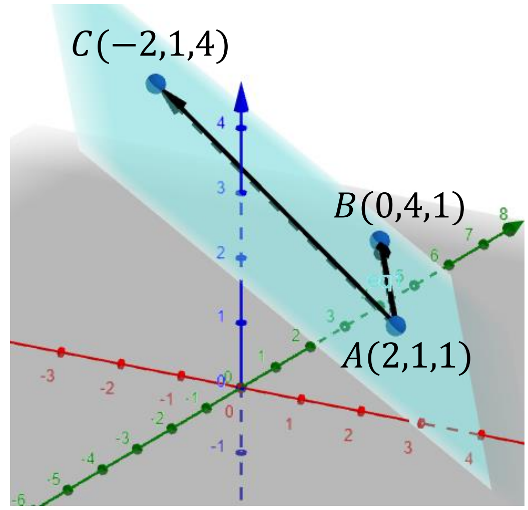
$\vec{n} \cdot \overrightarrow{P_0P} = \vec{n} \cdot < x - x_0, y - y_0, z - z_0 > = 0$

$A(x - x_0) + B(y - y_0) + B(z - z_0) = 0.$



Ex. Find an equation of a plane containing the points

$A(2,1,1),\ B(0,4,1),\ C(-2,1,4)$.



$\overrightarrow{AB} =< 0-2, 4-1, 1-1 > =< -2, 3, 0 >$

$\overrightarrow{AC} =< -2-2, 1-1, 4-1 > =< -4, 0, 3 >$.

$\overrightarrow{AB}$ and $\overrightarrow{AC}$ are vectors that lie in the plane containing

$A(2,1,1),\ B(0,4,1),\ C(-2,1,4)$.

How do we find a vector, $\vec{n}$, perpendicular to the plane containing A, B, C?

$$\vec{n} = \overrightarrow{AB} \times \overrightarrow{AC} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ -2 & 3 & 0 \\ -4 & 0 & 3 \end{vmatrix}$$

$$= \begin{vmatrix} 3 & 0 \\ 0 & 3 \end{vmatrix} \vec{i} - \begin{vmatrix} -2 & 0 \\ -4 & 3 \end{vmatrix} \vec{j} + \begin{vmatrix} -2 & 3 \\ -4 & 0 \end{vmatrix} \vec{k}$$

$$= 9\vec{i} - (-6)\vec{j} + \big(-(-12)\big)\vec{k} = 9\vec{i} + 6\vec{j} + 12\vec{k}.$$

$\vec{n} =< 9, 6, 12 >=< A, B, C >$.

Using any point on the plane we can take $(x_0, y_0, z_0) = (2,1,1)$.

An equation of the plane: $A(x - x_0) + B(y - y_0) + B(z - z_0) = 0$

$$9(x-2) + 6(y-1) + 12(z-1) = 0;$$

Or $$9x + 6y + 12z - 36 = 0$$

Or $$3x + 2y + 4z = 12.$$

4. Equations of Cylinders and a few common Quadric Surfaces in R^3

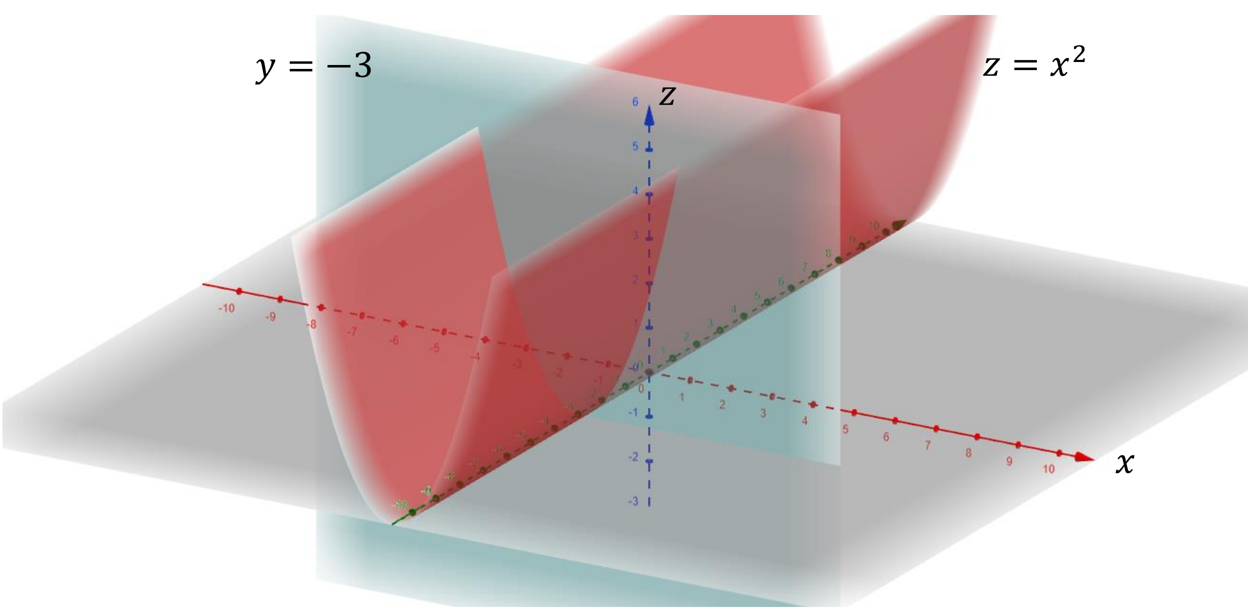
Def. A **cylinder** consists of all lines that are parallel to a given line and pass through a given plane curve.

Notice that if an equation in R^3 contains only 2 variables, the graph is a cylinder.

When picturing a graph of an equation in R^3 it is often helpful to examine the level curves ($z = constant$) and sections ($y = constant$ and $x = constant$) of the graph.

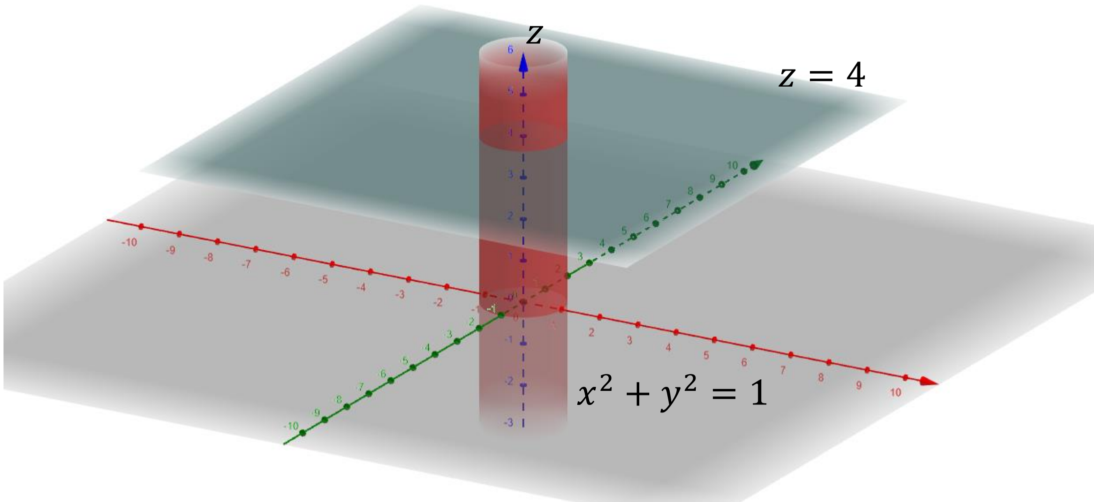
Ex. Sketch $z = x^2$ in R^3

In the xz plane ($y = 0$) this is just the parabola $z = x^2$. Since the function does not have a "y" in it, every cross sectional of the plane $y = k$ is the same parabola. This is called a parabolic cylinder. In fact, if one of x, y, z is missing from the equation, then you will get a cylinder.

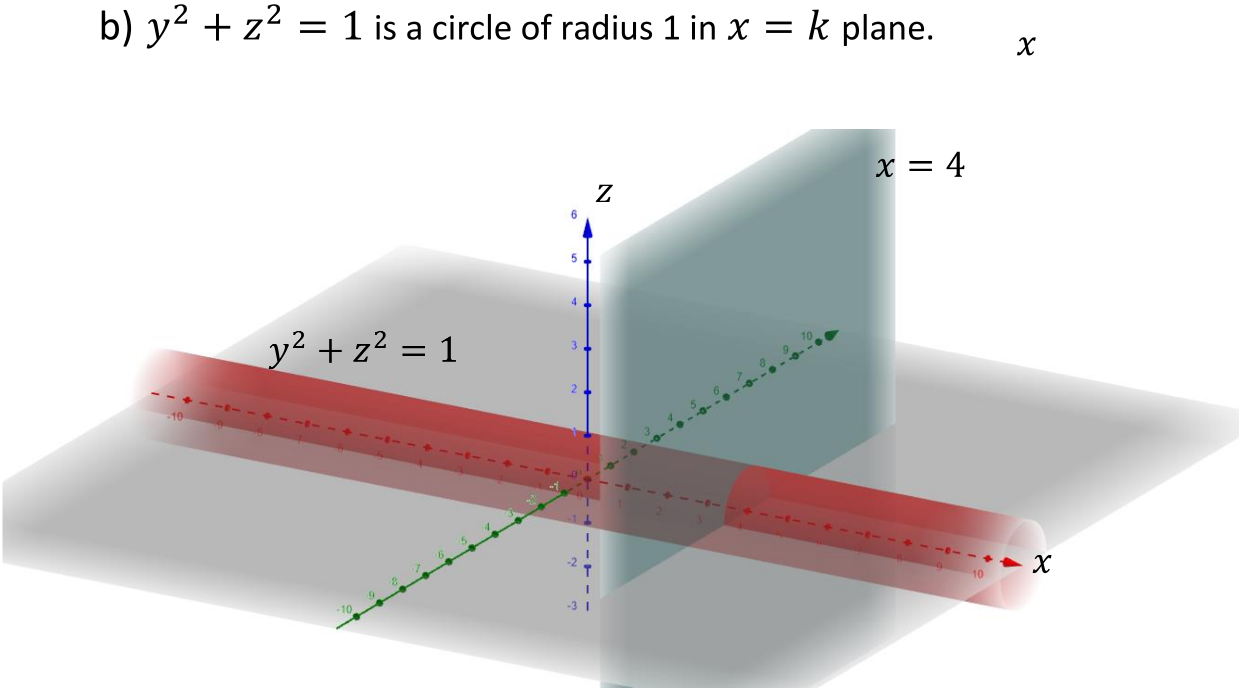


Ex. Sketch in $\mathbb{R}^3$: a) $x^2 + y^2 = 1$ b) $y^2 + z^2 = 1$

a) $x^2 + y^2 = 1$ is a circle of radius 1 in $z = k$ plane.



b) $y^2 + z^2 = 1$ is a circle of radius 1 in $x = k$ plane.



The graph of any equation of the form: $\frac{z^2}{a^2} = \frac{x^2}{b^2} + \frac{y^2}{c^2}$ is a cone.

Ex. sketch $z^2 = \frac{x^2}{2} + \frac{y^2}{3}$, using level curves and sections.

$z = k$: $k^2 = \frac{x^2}{2} + \frac{y^2}{3}$ slices ll to xy plane are ellipses if $k \neq 0$,

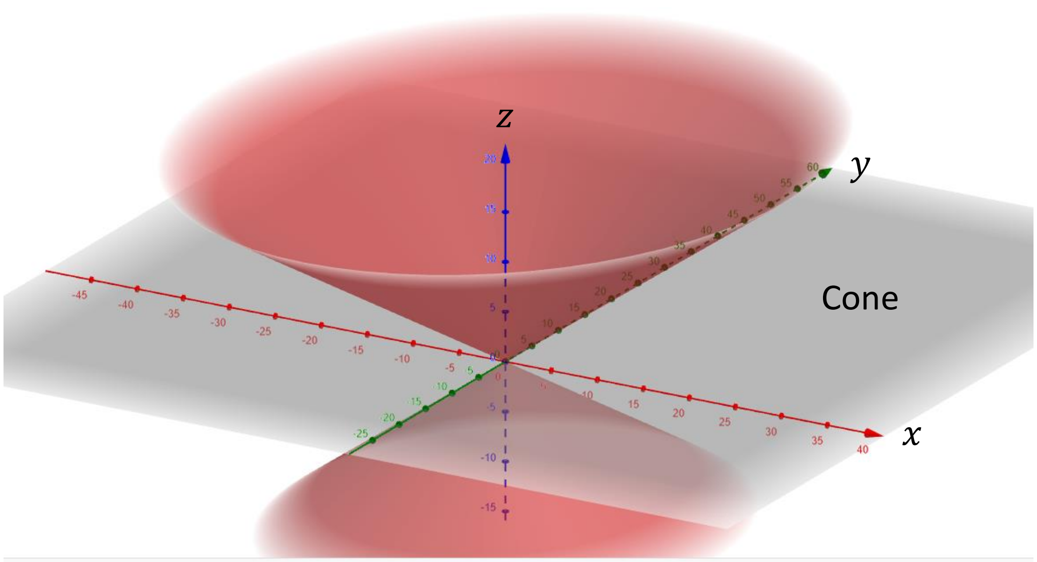
if $k = 0$, then it's a point.

$x = k$: $z^2 - \frac{y^2}{3} = \frac{k^2}{2}$ slices ll to yz plane are hyperbolas if $k \neq 0$,

major axis is the z axis.

$y = k$: $z^2 - \frac{x^2}{2} = \frac{k^2}{3}$ slices ll to xz plane are hyperbolas if $k \neq 0$,

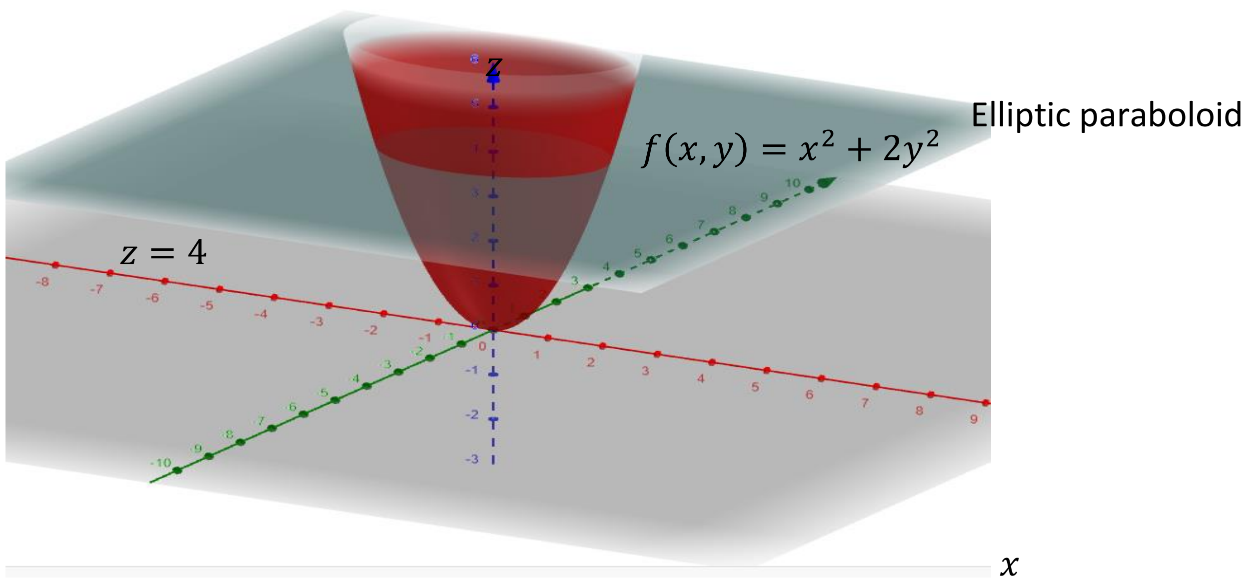
major axis is the z axis.



The graph of any equation of the form $\frac{z}{c} = \frac{x^2}{a^2} + \frac{y^2}{b^2}$ is an elliptic paraboloid.

Ex. Sketch $f(x,y) = x^2 + 2y^2$. Domain = $\mathbb{R}^2$; range $z \geq 0$.

Level curves for $z = k > 0$ are ellipses.



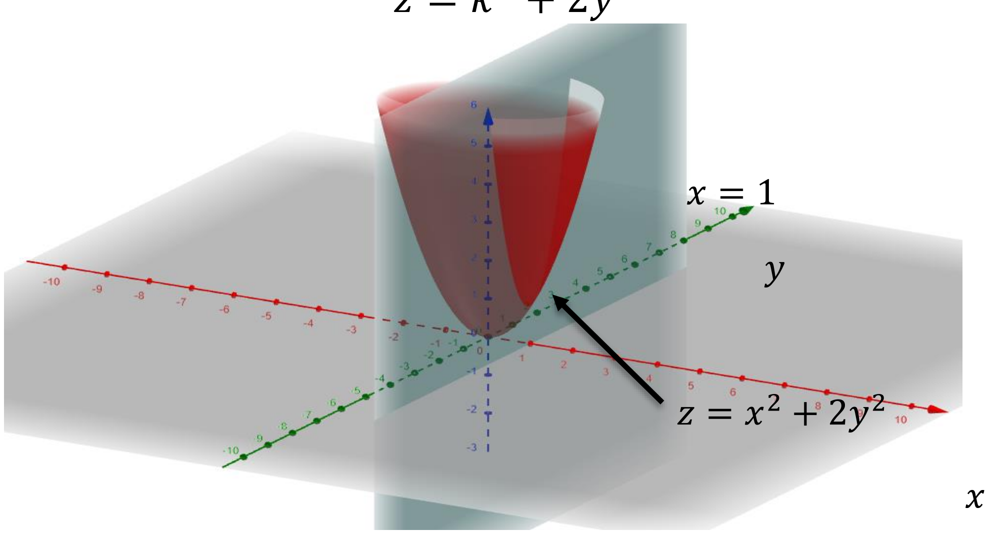
Sections of the graph of f are parabolas.

For example, if $y = k$, then we get:

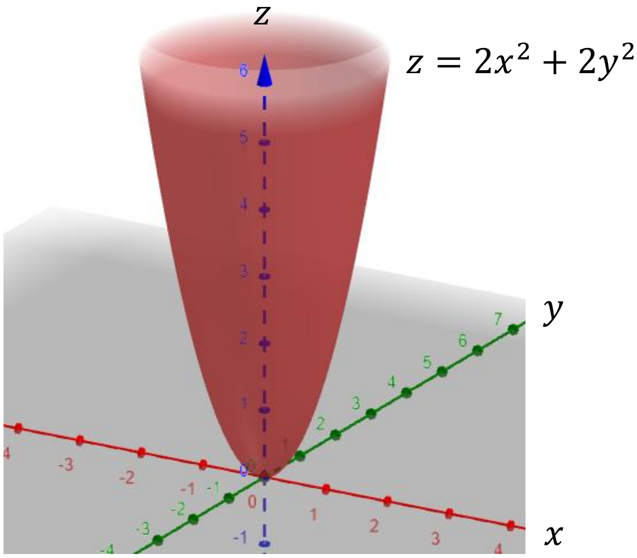
$$z = x^2 + 2k^2$$

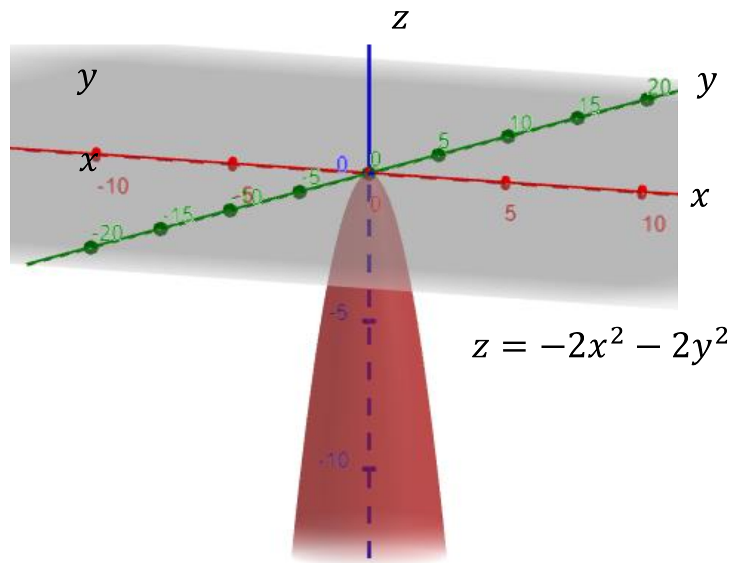
If $x = k$, then we get:

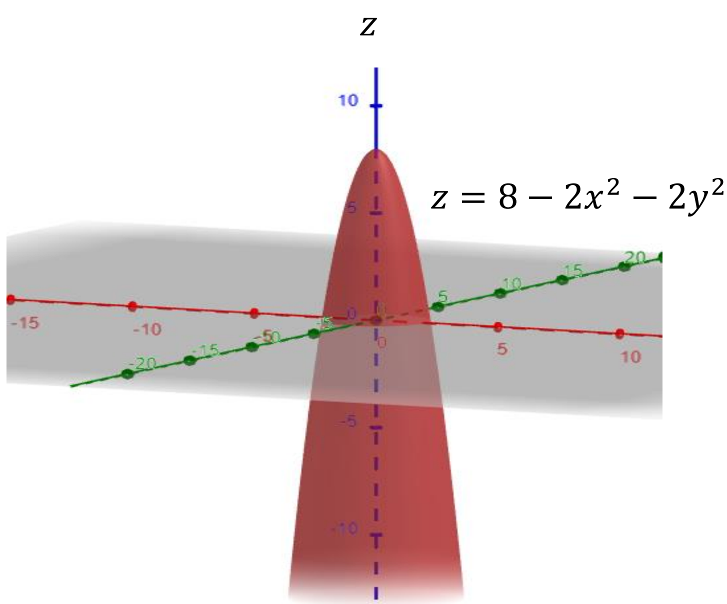
$$z = k^2 + 2y^2$$



Ex. Sketch a graph of $z = -2x^2 - 2y^2$ and $z = 8 - 2x^2 - 2y^2$.







The graph of any equation of the form $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$ is an ellipsoid (when $a=b=c$ you get a sphere).

Ex. Use the level curves and sections to sketch $x^2+\frac{y^2}{9}+\frac{z^2}{4}=1$.

When $z=0$: $x^2+\frac{y^2}{9}=1$ ellipse in the xy plane

$z=k$: $x^2+\frac{y^2}{9}+\frac{k^2}{4}=1$

$x^2+\frac{y^2}{9}=1-\frac{k^2}{4}$ is an ellipse if $-2<k<2$. As $k\to 2$ or -2 the major and minor axes are shrinking to 0. For example,

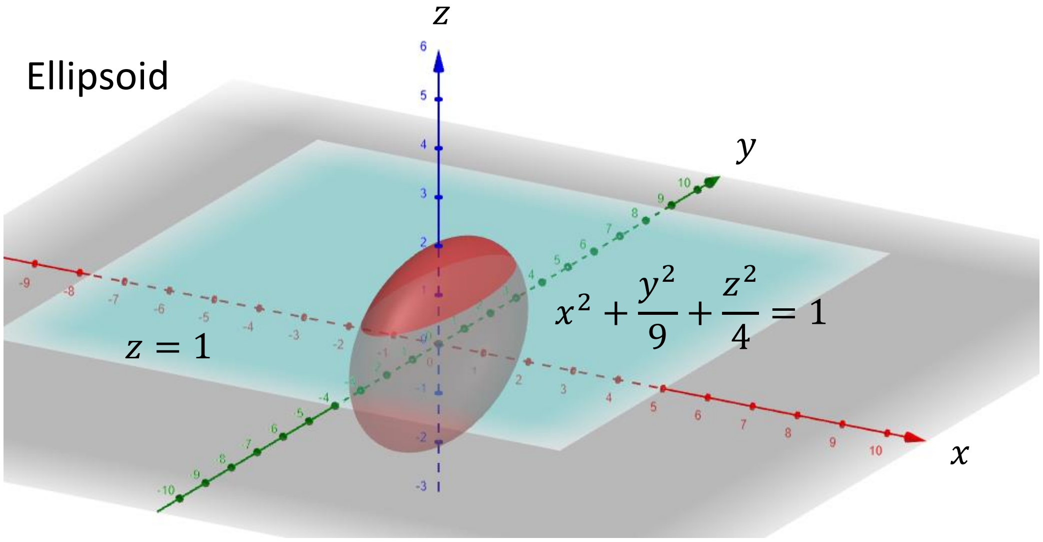
$$k=1:\ x^2+\frac{y^2}{9}=\frac{3}{4}\Rightarrow\frac{x^2}{\frac{3}{4}}+\frac{y^2}{\frac{27}{4}}=1.$$

At $x=0$: $\frac{y^2}{9}+\frac{z^2}{4}=1$ ellipse in yz plane

At $x=k$: $\frac{y^2}{9}+\frac{z^2}{4}=1-k^2$ $\quad -1<k<1$ ellipse

At $y=0$: $x^2+\frac{z^2}{4}=1$ ellipse in the xz plane

At $y=k$: $x^2+\frac{z^2}{4}=1-\frac{k^2}{9}$ $\quad -3<k<3$ ellipse

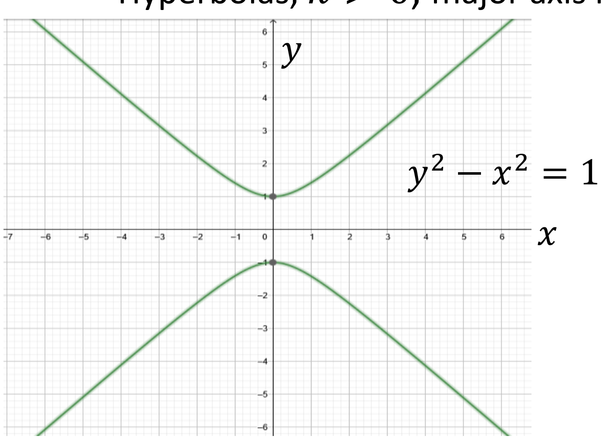


The graph of $z = \frac{x^2}{a^2} - \frac{y^2}{b^2}$ or $z = \frac{y^2}{a^2} - \frac{x^2}{b^2}$ is a hyperbolic paraboloid (ie a saddle).

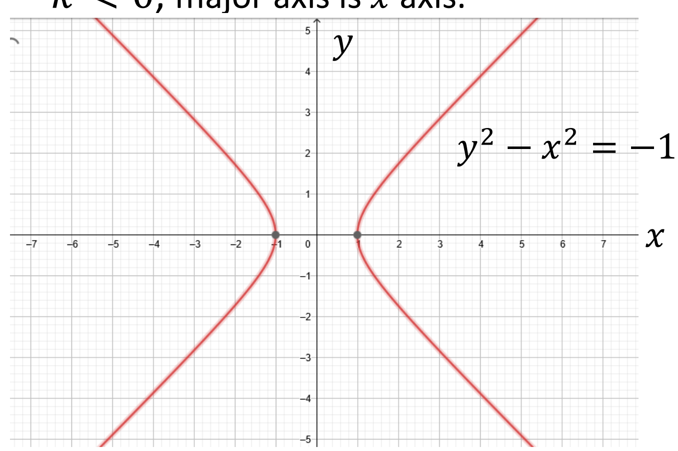
Ex. Use level curves and sections to sketch $z = y^2 - x^2$.

Level curves: $k = y^2 - x^2$:

Hyperbolas, $k > 0$, major axis is y axis



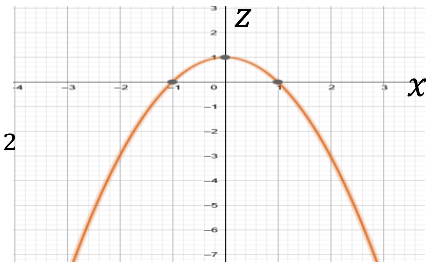
$k < 0$, major axis is x axis.



Sections:

$k = 1$:

$z = 1 - x^2$



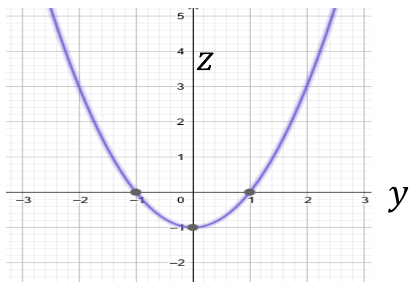
$y = k; \quad z = k^2 - x^2$

parabolas in xz plane opening

in negative z direction.

$k = 1$:

$z = y^2 - 1$

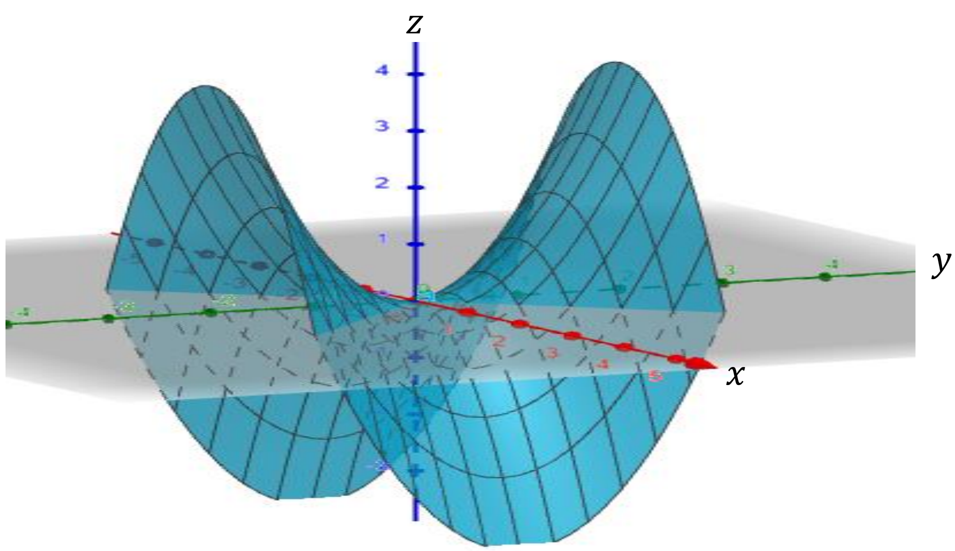


$x = k; \quad z = y^2 - k^2$

parabolas in yz plane opening

opening in positive z direction.

Hyperbolic Paraboloid (Saddle)



The graph of $\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1$ is a hyperboloid of one sheet.

Ex. Sketch $\frac{x^2}{4} + y^2 - \frac{z^2}{4} = 1$, using level curves and sections.

At $z = k$: $\frac{x^2}{4} + y^2 = 1 + \frac{k^2}{4}$ ellipse in slices II xy plane

At $y = 0$: $\frac{x^2}{4} - \frac{z^2}{4} = 1$ hyperbola in slices II xz plane

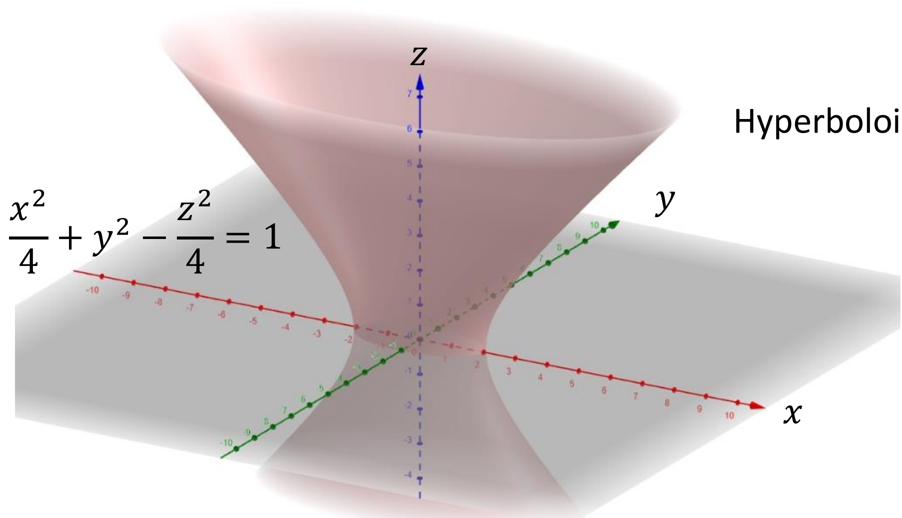
At $y = k$: $\frac{x^2}{4} - \frac{z^2}{4} = 1 - k^2$, hyperbolas in xz plane if $k \neq \pm 1$

$-1 < k < 1$, major axis is x axis; $k < -1$ or $k > 1$, major axis is z axis.

At $x = 0$: $y^2 - \frac{z^2}{4} = 1$ hyperbolas in slices II to yz plane

At $x = k$: $y^2 - \frac{z^2}{4} = 1 - \frac{k^2}{4}$, hyperbolas in yz plane if $k \neq \pm 2$

$-2 < k < 2$, major axis is y axis; $k < -2$ or $k > 2$, major axis is z axis.



Hyperboloid of one sheet

If $-\frac{x^2}{4} + y^2 + \frac{z^2}{4} = 1$, then the major axis is the x-axis (with negative term).