The Riemann Integral

Let f be a bounded real valued function on $[a, b]$ (a, b finite). Let $P = \{x_0, x_1, \dots, x_n\}$ be a partition of $[a, b]$:

$$a = x_0 < x_1 < x_2 < \cdots < x_n = b.$$

We define the lower and upper **Darboux sums** for f and P by:

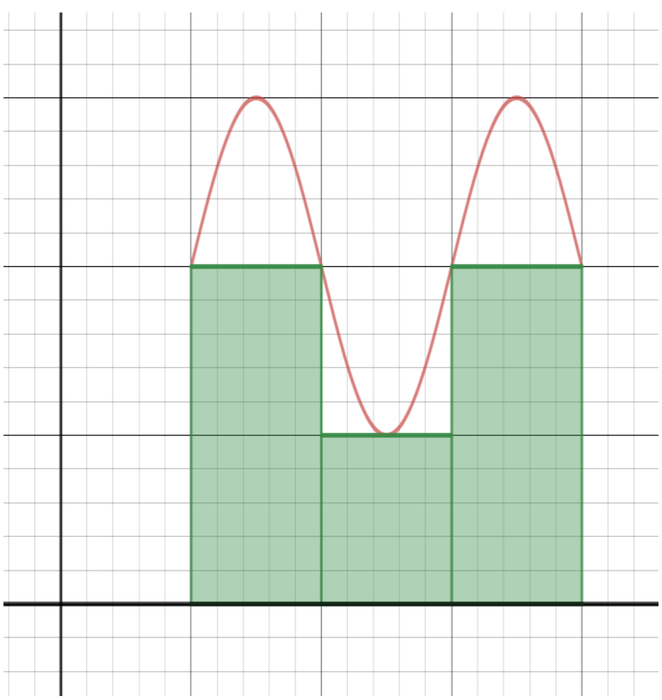
$$L(f, P) = \sum_{i=1}^{n} m_i (x_i - x_{i-1})$$

$$U(f, P) = \sum_{i=1}^{n} M_i (x_i - x_{i-1})$$

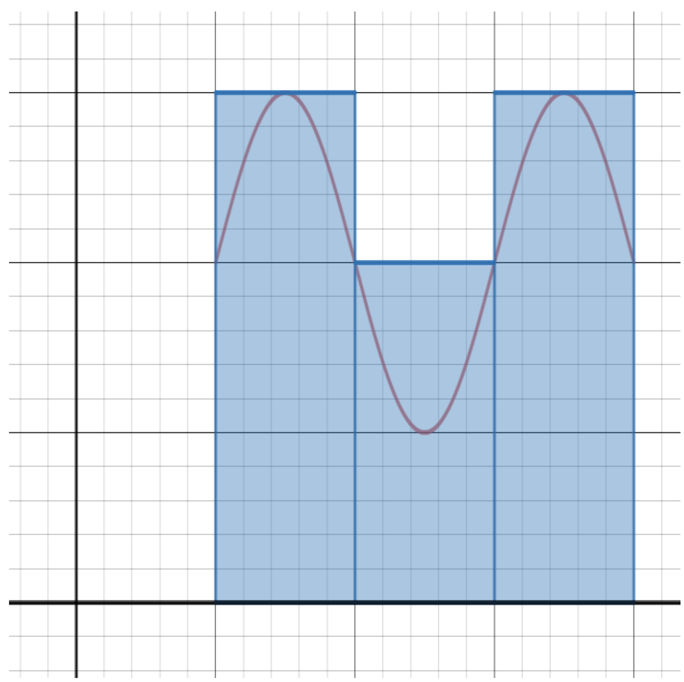
where $m_i = \inf \{f(x) |\ x_{i-1} < x < x_i\}$

$$M_i = \sup \{f(x) |\ x_{i-1} < x < x_i\}.$$

Lower Darboux Sum



Upper Darboux Sum



We define the **lower and upper Riemann sums of $\boldsymbol{f}$** over $[a, b]$ as:

$$\underline{\int_a^b} f = \sup\{L(f, P) |\ P\ a\ partition\ of\ [a, b]\}$$

$$\overline{\int_a^b} f = \inf\{U(f, P) |\ P\ a\ partition\ of\ [a, b]\}.$$

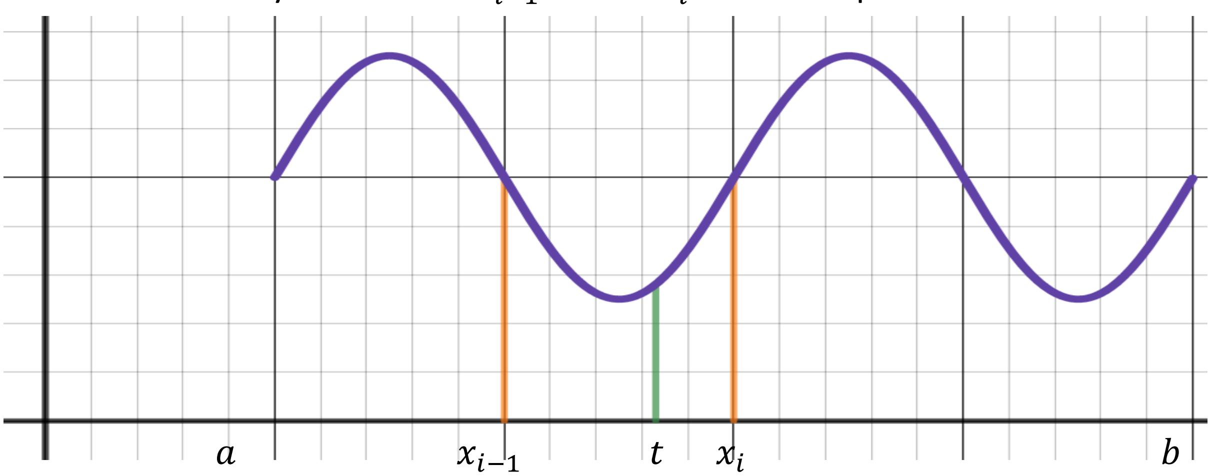
Since f is bounded and $[a, b]$ has finite length, $L(f, P) \leq U(f, P)$ and

$\underline{\int_a^b} f \leq \overline{\int_a^b} f$.

If $\underline{\int_a^b} f = \overline{\int_a^b} f$ we say that f is **Riemann integrable over $\boldsymbol{[a, b]}$.**

Proposition: If P' is a refinement of P (i.e. P' contains all of the points of P plus others) then $L(f, P') \geq L(f, P)$ and $U(f, P') \leq U(f, P)$.

Proof. Choose any subinterval $x_{i-1} \leq x \leq x_i$ and add a point t.



Let $m_i' = \inf_{x_{i-1} \leq x \leq t} f(x)$ and $m_i'' = \inf_{t \leq x \leq x_i} f(x)$.

Then $m_i' \geq m_i$ and $m_i'' \geq m_i$.

Now just using the interval $x_{i-1} \le x \le x_i$ we have:

$$
\begin{aligned}
L(f,P') &= m_i'(t - x_{i-1}) + m_i''(x_i - t) \\
&\ge m_i(t - x_{i-1}) + m_i(x_i - t) \\
&= m_i(x_i - x_{i-1}) = L(f,P).
\end{aligned}
$$

A similar argument shows $U(f,P') \le U(f,P)$.

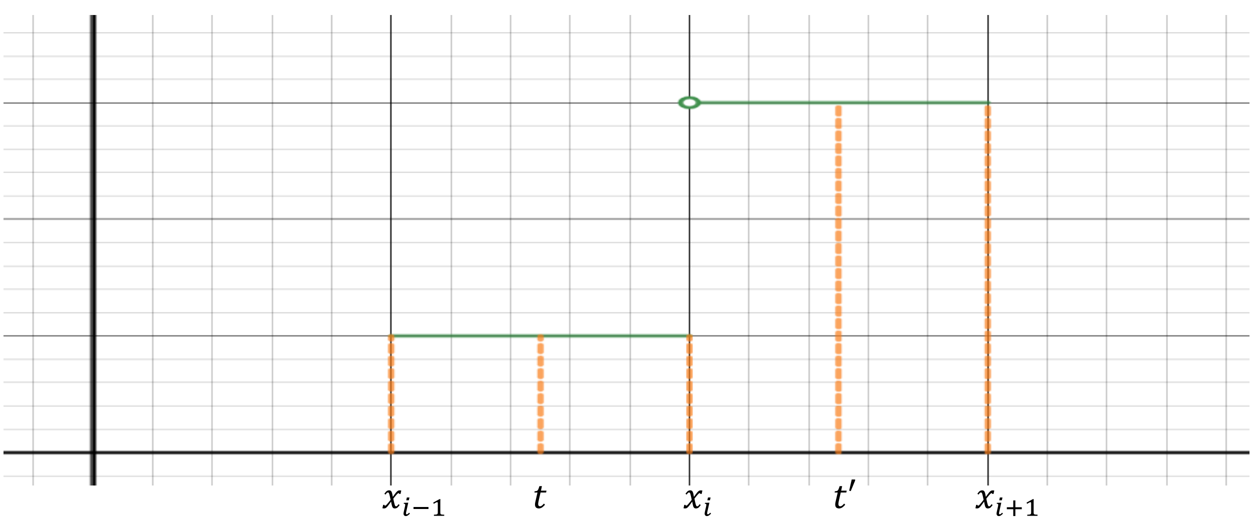
Def. A real valued function ψ defined on $[a,b]$ is called a **step function** provided there is a partition $P = \{x_0, x_1, \dots, x_n\}$ of $[a,b]$ and numbers

$c_1, c_2, \dots, c_n$ such that for $1 \le i \le n$; $\psi(x) = c_i$ if $x_{i-1} < x < x_i$.

Notice that for a step function ψ and any partition Q,

$$L(\psi, Q) = \sum_{i=1}^{n} c_i(x_i - x_{i-1}) = U(\psi, Q).$$

Thus we get $\int_a^b \psi = \sum_{i=1}^{n} c_i(x_i - x_{i-1})$.



We can now reformulate the definition of the **upper and lower Riemann sum** as:

$$\underline{\int_a^b} f = \sup\left\{\underline{\int_a^b} \varphi \,\middle|\, \varphi \text{ a step function and } \varphi \le f \text{ on } [a,b]\right\}$$

$$\overline{\int_a^b} f = \inf\left\{\overline{\int_a^b} \varphi \mid \varphi \text{ a step function and } \varphi \ge f \text{ on } [a,b]\right\}.$$

Ex. Let $f(x) = 1$ if $x \in \mathbb{Q} \cap [0,1]$

$= 0$ if $x \notin \mathbb{Q} \cap [0,1]$.

Let P be any partition of $[0,1]$, then

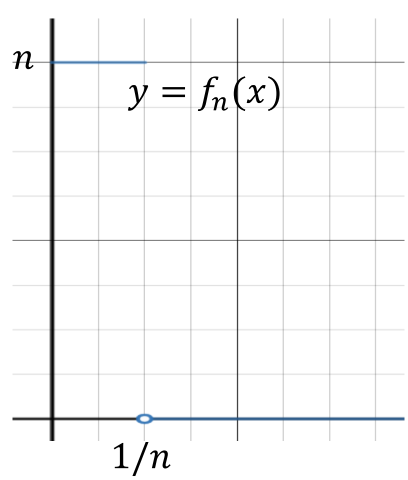
$L(f, P) = 0$ and $U(f, P) = 1$.

Thus $\underline{\int_0^1} f = 0$ and $\overline{\int_0^1} f = 1$.

Hence f is not Riemann integrable on $[0,1]$.

So far we know of two ways in which a sequence of function $\{f_n\}$ can converge to a function f, pointwise and uniformly. We are going to be interested in other notions of convergence. These will be related to integration.

Ex. If $\{f_n\} \to f$ pointwise on $[a, b]$ and f_n and f are Riemann integrable over $[a, b]$, is it true that $\lim_{n\to\infty} \int_a^b f_n = \int_a^b f$?



No! Here's an example.

Let $f_n(x) = 0$ if $\frac{1}{n} < x \le 1$ or $x = 0$

$= n$ if $0 < x \le \frac{1}{n}$.

Notice that $\lim_{n\to\infty} f_n = f$ where $f(x) = 0$ for all $0 \le x \le 1$.

That is, $\{f_n\} \to f$ pointwise on $x \in [0,1]$.

However, $\int_0^1 f_n = 1$ for all n so

$\lim_{n\to\infty} \int_0^1 f_n = 1,$ but $\int_0^1 f = 0.$

So $\lim_{n\to\infty} \int_0^1 f_n \ne \int_0^1 f.$