The Simple Approximation Theorem

Def. Given a sequence of functions $\{f_n\}$ with common domain E, a function f on E and a subset $G \subseteq E$, we say:

1) The sequence $\{\boldsymbol{f_n}\}$ **converges to $\boldsymbol{f}$ pointwise on $\boldsymbol{G}$** if $\lim_{n\to\infty} f_n(x) = f(x)$ for all $x \in G$.

2) The sequence $\{\boldsymbol{f_n}\}$ **converges to $\boldsymbol{f}$ pointwise a.e. on $\boldsymbol{G}$** if it converges to f pointwise on $G \sim H$, where $m(H) = 0$.

3) The sequence $\{\boldsymbol{f_n}\}$ **converges to $\boldsymbol{f}$ uniformly on $\boldsymbol{G}$** if for each $\epsilon > 0$, there is an N such that if $n \geq N$ then:

$$|f(x) - f_n(x)| < \epsilon \text{ on } G$$

The pointwise limit of continuous functions need not be continuous.

Ex. In the example below prove that $f_n \to f$ pointwise on $0 \leq x \leq 1$ but not uniformly.

$$f_n(x) = x^n; \quad 0 \leq x \leq 1.$$

$$f(x) = 0 \quad \text{if } 0 \leq x < 1$$

$$= 1 \quad \text{if } x = 1$$

To prove pointwise convergence we must show that given any $\epsilon > 0$ there exists an N such that if $n \geq N$ then $|f(x) - f_n(x)| < \epsilon$.

Note: for pointwise convergence N can depend on the point x.

At $x = 0$, $f_n(0) = 0 = f(0)$ for all n, thus $f_n(0) \to f(0)$.

If $0 < x < 1$, then $f(x) = 0$.

$|f(x) - f_n(x)| = |0 - x^n| = x^n;$ since $0 < x < 1$.

So solve $x^n < \epsilon$ for n.

$$n(lnx) < ln\epsilon$$

$$n > \frac{ln\epsilon}{\ln(x)}.$$

So choose $N > \frac{ln\epsilon}{\ln(x)}$.

Notice our formula for N depends on the point x.

So if $N > \frac{ln\epsilon}{\ln(x)}$ and $n \geq N > \frac{ln\epsilon}{\ln(x)}$

Then $n > \frac{ln\epsilon}{\ln(x)}$

$$n(lnx) < ln\epsilon$$

$$x^n < \epsilon$$

$$|0 - x^n| < \epsilon.$$

So $f_n \to f$ pointwise on $0 \leq x < 1$.

At $x = 1$, $f_n(1) = 1 = f(1)$ for all n so $f_n \to f$ pointwise on $0 \leq x \leq 1$.

Notice that $f_n(x)$ does not converge uniformly to $f(x)$ on $[0,1]$ because:

$|0 - x^n| < \epsilon$ is equivalent to $n > \frac{ln\epsilon}{\ln(x)}$, and as $x \to 1$, $\frac{ln\epsilon}{\ln(x)}$ is unbounded above for any fixed ϵ.

Thus there isn't any fixed N such that $n \geq N \Rightarrow |0 - x^n| < \epsilon$ for all $0 \leq x \leq 1$.
Thus $f_n(x)$ doesn't converges uniformly to $f(x)$ for $0 \leq x \leq 1$.

Note: $f_n(x)$ converges uniformly to $f(x)$ for $0 \leq x \leq 1 - \alpha$, for any $0 < \alpha < 1$.

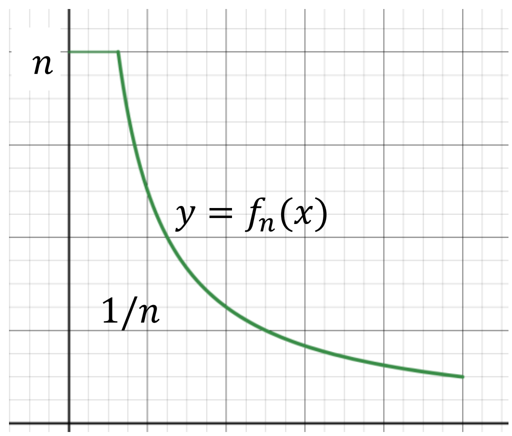
This is because $\frac{ln\epsilon}{\ln(x)} \leq \frac{ln\epsilon}{\ln(1-\alpha)}$, for all $0 \leq x \leq 1 - \alpha$, and $0 < \epsilon \leq 1$.

So $N > \frac{ln\epsilon}{ln\alpha}$ would work for all $0 \leq x \leq 1 - \alpha$.

The pointwise limit of Riemann integrable functions may not be Riemann integrable.

Ex. Let $f_n(x) = n$ if $0 < x \leq \frac{1}{n}$

$= \frac{1}{x}$ if $\frac{1}{n} < x \leq 1$.



$\{f_n\}$ converges pointwise to $f(x) = \frac{1}{x}$ on $0 < x \leq 1$ (which is not Riemann integrable).

Note: The convergence is not uniform. For example, if $\epsilon = 1$ and

$0 < x < \frac{1}{n}$; $|f_n(x) - f(x)| = |n - \frac{1}{x}|$, which is unbounded for all n.

However, we do have:

Prop. Let $\{f_j\}$ be a sequence of measurable functions on E that converges pointwise a.e. on E to a function f. Then f is measurable.

Proof: Suppose $\{f_j\}$ converges pointwise to f on $E{\sim}A$, where $m(A) = 0, A \subseteq E$.

From an earlier proposition we know that f is measurable if, and only if, its restriction to $E{\sim}A$ is measurable.

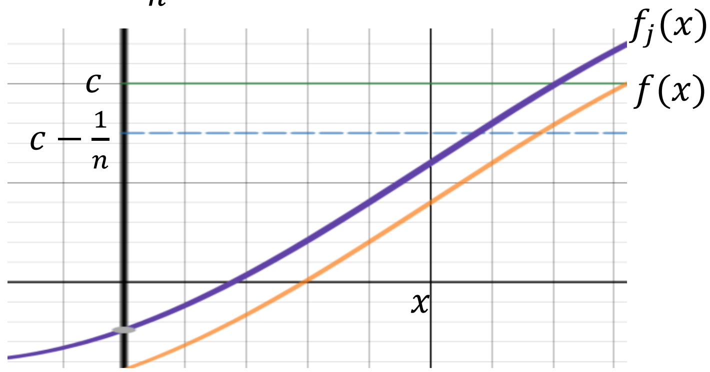
Thus, by replacing E by $E{\sim}A$, we can assume the sequence converges pointwise on E.

We must show $\{x \in E | f(x) < c\}$ is measurable.

For a point $x \in E$, since $\lim_{j \to \infty} f_j(x) = f(x)$,

$f(x) < c$ if, and only if, there exist $n, k \in \mathbb{Z}^+$ such that:

$f_j(x) < c - \frac{1}{n}$ for all $j \geq k$, where k depends on x and n.



But for any $n, j \in \mathbb{Z}^+$, $\{x \in E |\ f_j(x) < c - \frac{1}{n}\}$ is measurable since f_j is measurable.

Thus, for any k,

$$\bigcap_{j=k}^{\infty}\{x \in E \mid f_j(x) < c - \tfrac{1}{n}\} = \left\{x \in E \,\middle|\, f(x) < c - \tfrac{1}{n}\right\}$$

is also measurable.

Now notice:

$$\{x \in E | f(x) < c\} = \bigcup_{1 \le k,n < \infty}[\bigcap_{j=k}^{\infty}\{x \in E \mid f_j(x) < c - \tfrac{1}{n}\}].$$

So $f(x)$ is measurable because the RHS is made up of countable unions and intersections of measurable sets.

Def. If A is any set, the **characteristic function** of A, χ_A, is the function on $\mathbb{R}$ defined by:

$$\chi_A(x) = 1 \text{ if } x \in A$$
$$= 0 \text{ if } x \notin A.$$

χ_A is measurable if, and only if, A is measurable. Linear combinations of characteristic functions will play a role in Lebesgue integration.

Def. A real valued function φ on a measurable set E is called **simple** if it is measurable and takes on only a finite number of values.

Notice that linear combinations and products of simple functions are simple functions.

If φ is a simple function on a domain E that takes the values $c_1, c_2, \dots, c_n$ then we can write φ as:

$$\varphi(x) = \sum_{k=1}^{n} c_k\, \chi_{E_k}(x); \text{ where } E_k = \{x \in E | \varphi(x) = c_k\}.$$

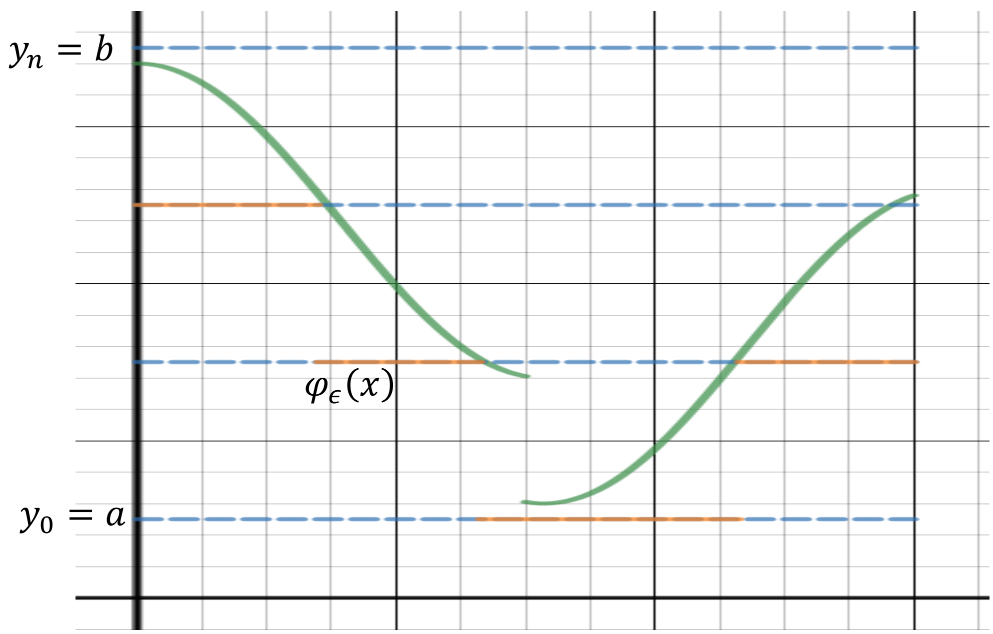
This representation of a simple function as a linear combination of characteristic functions is called the **canonical representation** of the simple function φ.

The Simple Approximation Lemma: Let f be a measurable real valued function on E. Assume f is bounded on E, i.e. there is an $M \geq 0$ for which $|f(x)| \leq M$ for all $x \in E$. Then for each $\epsilon > 0$ there are simple functions φ_ϵ and ψ_ϵ defined on E such that:

$$\varphi_\epsilon(x) \leq f(x) \leq \psi_\epsilon(x) \text{ and } 0 \leq \psi_\epsilon(x) - \varphi_\epsilon(x) < \epsilon \text{ on } E.$$

Proof: Let (a, b) be an open, bounded interval that contains the image of $E, f(E)$, and $a = y_0 < y_1 < \cdots < y_{n-1} < y_n = b$ where

$y_j - y_{j-1} < \epsilon$ for $1 \leq j \leq n$.



Let $I_j = [y_{j-1}, y_j)$ and $E_j = f^{-1}(I_j)$

Since f is measurable and I_j is measurable, each E_j is measurable.

Define $\varphi_\epsilon = \sum_{j=1}^{n} y_{j-1}\, \chi_{E_j}$ and $\psi_\epsilon = \sum_{j=1}^{n} y_j\, \chi_{E_j}$.

Let $x \in E$, since $f(E) \subseteq (a,b)$, there is a unique $j, 1 \le j \le n$ for which $y_{j-1} \le f(x) < y_j$ and therefore:

$$\varphi_\epsilon(x) = y_{j-1} \le f(x) < y_j = \psi_\epsilon(x)$$

But $y_j - y_{j-1} < \epsilon$ so, $0 \le \psi_\epsilon(x) - \varphi_\epsilon(x) < \epsilon$.

Ex. Let $f(x) = x^2$ if $-2 < x < 2$ and $x \neq 0$

$= 2$ if $x = 0$.

Approximate $f(x)$ by simple functions φ, ψ where:

$\varphi \le f \le \psi$ and $0 \le \psi - \varphi < 1.1$ on $(-2, 2)$.

Notice $0 < f(x) < 4$, so we need a partition of $[0, 4]$ such that each interval has length less than 1.1

One way to do this is: $a = 0 < 1 < 2 < 3 < 4 < b$

So $I_1 = [0, 1)$

$I_2 = [1, 2)$

$I_3 = [2, 3)$

$I_4 = [3, 4)$

$\varphi_\epsilon = \sum_{j=1}^{4} y_{j-1}\, \chi_{E_j}$ and $\psi_\epsilon = \sum_{k=1}^{4} y_j\, \chi_{E_j}$

where $E_j = f^{-1}(I_j)$ $1 \le j \le 4$.

$$E_1 = f^{-1}(I_1) = \{x|\, 0 \le f(x) < 1\} = \{x|\, 0 \le x^2 < 1, x \ne 0\}$$
$$= (-1, 0) \cup (0,1)$$

$$E_2 = f^{-1}(I_2) = \{x|\, 1 \le f(x) < 2\} = \{x|\, 1 \le x^2 < 2, x \ne 0\}$$
$$= \left(-\sqrt{2}, -1\right] \cup \left[1, \sqrt{2}\right)$$

$$E_3 = f^{-1}(I_3) = \{x|\, 2 \le f(x) < 3\} = \{x|\, 2 \le x^2 < 3\} \cup \{0\}$$
$$= \left(-\sqrt{3}, -\sqrt{2}\right] \cup \left[\sqrt{2}, \sqrt{3}\right) \cup \{0\}$$

$$E_4 = f^{-1}(I_4) = \{x|\, 3 \le f(x) < 4\} = \left(-2, -\sqrt{3}\right] \cup [\sqrt{3}, 2).$$

So

$$\varphi_{1.1}(x) = 0 \cdot \chi_{E_1} + 1 \cdot \chi_{E_2} + 2 \cdot \chi_{E_3} + 3 \cdot \chi_{E_4} = \chi_{E_2} + 2\chi_{E_3} + 3\chi_{E_4}$$
$$\psi_{1.1}(x) = \chi_{E_1} + 2\chi_{E_2} + 3\chi_{E_3} + 4\chi_{E_4}.$$

The Simple Approximation Theorem: An extended real valued function f on a measurable set E is measurable if, and only if, there is a sequence $\{\varphi_n\}$ of simple functions on E which converges pointwise to f on E and has $|\varphi_n(x)| \le |f(x)|$ on E for all n.

If f is non-negative, we may choose $\{\varphi_n\}$ to be increasing.

Proof: Since each simple function is measurable, and we know from an earlier proposition that the pointwise limit of measurable functions is measurable, f is then measurable.

Now assume f is measurable and let's show we can find a sequence of of simple functions that converges pointwise to it on E.

First let's assume $f \geq 0$ on E. Let $E_n = \{x \in E | \, f(x) \leq n\}$.

Then E_n is measurable and the restriction of f to E_n is a non-negative bounded measurable function. By the previous lemma applied to E_n and with $\epsilon = \frac{1}{n}$ we can find simple functions $\varphi_\epsilon, \psi_\epsilon$:

$$0 \leq \varphi_n \leq f \leq \psi_n \text{ on } E_n \quad \text{and} \quad 0 \leq \psi_n - \varphi_n < \frac{1}{n} \text{ on } E_n.$$

Also:

$$0 \leq \varphi_n \leq f \quad \text{and} \quad 0 \leq f - \varphi_n \leq \psi_n - \varphi_n < \quad \text{on } E_n.$$

We can extend φ_n to all of E by setting $\varphi_n(x) = n$ if $f(x) > n$.
Now $0 \leq \varphi_n \leq f$ on all of E.

Now let's show $\lim_{n\to\infty} \varphi_n(x) = f(x)$ for $x \in E$.

Case 1: Assume $f(x)$ is finite.

Choose $N \in \mathbb{Z}^+$ such that $f(x) < N$.

Then, $0 \leq f(x) - \varphi_n(x) < \frac{1}{n}$ for $n \geq N$.

Since if $n \geq N$ and $f(x) < N$, then $E_n = E$.

Thus, $\lim_{n\to\infty} \varphi_n(x) = f(x)$.

Case 2: $f(x) = \infty$.

Then $\varphi_n(x) = n$ for all n, so $\lim_{n\to\infty} \varphi_n(x) = f(x)$.

By replacing each φ_n with max $\{\varphi_1(x), \dots, \varphi_n(x)\}$ we get $\{\varphi_n\}$ increasing.

The general case follows by expressing f by:

$$f(x) = f^+(x) - f^-(x)$$

where $f^+(x)$ and $f^-(x)$ are both non-negative functions.