Lebesgue Measurable Functions

We will assume all functions have domains that are a subset of $\mathbb{R}$ and take values in $\mathbb{R} \cup \{\pm\infty\}$.

Prop. Let f have a measurable domain E. Then the following statements are equivalent.

1. For each $c \in \mathbb{R}$, the set $\{x \in E | f(x) > c\}$ is measurable.
2. For each $c \in \mathbb{R}$, the set $\{x \in E | f(x) \geq c\}$ is measurable.
3. For each $c \in \mathbb{R}$, the set $\{x \in E | f(x) < c\}$ is measurable.
4. For each $c \in \mathbb{R}$, the set $\{x \in E | f(x) \leq c\}$ is measurable.

Each of these properties implies that for each extended real number c, the set $\{x \in E | f(x) = c\}$ is measurable.

Proof: Sets 1 and 4 , and 2 and 3, are complements . Since complements of measurable sets are measurable 1 and 4 are equivalent and 2 and 3 are equivalent.

1⇒2.

$$\{x \in E | f(x) \geq c\} = \bigcap_{n=1}^{\infty} \left\{x \in E \middle| f(x) > c - \frac{1}{n}\right\}.$$

By 1, each set $\left\{x \in E \middle| f(x) > c - \frac{1}{n}\right\}$ is measurable.

The countable intersection of measurable sets is measurable hence

$\{x \in E | f(x) \geq c\}$ is measurable.

2⇒1.

$$\{x \in E | f(x) > c\} = \cup_{n=1}^{\infty} \left\{x \in E \middle| f(x) \geq c + \frac{1}{n}\right\}.$$

By 2, each $\left\{x \in E \middle| f(x) \geq c + \frac{1}{n}\right\}$ is measurable hence so is $\{x \in E | f(x) > c\}$.

Thus statements 1-4 are equivalent.

Notice that if $c \in \mathbb{R}$ then

$$\{x \in E | f(x) = c\} = \{x \in E | f(x) \geq c\} \cap \{x \in E | f(x) \leq c\},$$

thus $\{x \in E | f(x) = c\}$ is measurable because it's the intersection of two measurable sets.

Notice that if $c = \infty$ then:

$$\{x \in E | f(x) = \infty\} = \cap_{n=1}^{\infty} \{x \in E | f(x) > n\}.$$

Thus $\{x \in E | f(x) = \infty\}$ is the countable intersection of measurable sets and hence measurable.

Def. An extended real valued function defined on E is said to be **Lebesgue measurable** (or just measurable), provided its domain E is measurable and it satisfies one (and hence all) of the four statements in the previous proposition.

Prop. Let $f: E \to \mathbb{R} \cup \{\pm\infty\}$, where E is measurable. Then f is measurable if and only if for each open set O, the inverse image of O, $f^{-1}(O) = \{x \in E | \; f(x) \in O\}$, is a measurable set.

Proof: $\Longrightarrow$ If $f^{-1}(O)$ is measurable for every open set O, then $f^{-1}(c,\infty) = \{x | f(x) > c\}$ is measurable and hence f is measurable.

$\Longleftarrow$ If f is measurable and O is any open set, then we can express O as the countable union of bounded, open intervals $\{I_n\}_{n=1}^{\infty}$, where

each I_n can be expressed as $B_n \cap A_n,\ B_n = (-\infty, b_n),\ A_n = (a_n, \infty)$.

Since f is measurable so are $f^{-1}(B_n)$ and $f^{-1}(A_n)$.

$$\begin{aligned} f^{-1}(O) &= f^{-1}(\cup_{n=1}^{\infty}(B_n \cap A_n)) \\ &= \cup_{n=1}^{\infty}(f^{-1}(B_n) \cap f^{-1}(A_n)). \end{aligned}$$

Since measurable sets form a σ-algebra, $f^{-1}(O)$ is measurable.

Prop. If $f: E \to \mathbb{R}$, where f is continuous and E is measurable, then f is measurable.

Proof: Since f is continuous, given any open set O in $\mathbb{R}$,

$f^{-1}(O) = E \cap U$, where U is open in $\mathbb{R}$.

Thus $f^{-1}(O)$ is measurable because E and U are.

Hence f is measurable by the previous proposition.

Def. A function that is either increasing on E or decreasing on E is called **monotonic**.

Prop. A monotonic function that is defined on an interval is measurable.

(HW problem).

Prop. Let f be an extended real valued function on a measurable set E.

1. If f is measurable on E and $f(x) = g(x)$ almost everywhere (a.e.), then g is measurable on E.
2. For a measurable subset $B \subseteq E$, f is measurable on E if and only if the restriction of f to B and $E \sim B$ are measurable.

Proof: 1. Assume f is measurable.

Let $F = \{x \in E | \; f(x) \neq g(x)\}$.

Notice that:

$\{x \in E | g(x) > c\} = \{x \in F | \; g(x) > c\} \cup (\{x \in E | \; f(x) > c\} \cap (E \sim F))$.

Since $f = g$ a.e., $m(F) = 0$ and hence $m\{x \in F | \; g(x) > c\} = 0$.

Thus F and $\{x \in F | \; g(x) > c\}$ are measurable.

Since f is measurable, $\{x \in E | \; f(x) > c\}$ is measurable.

$E \sim F$ is measurable because E and F are.

Thus $\{x \in E | g(x) > c\}$ is measurable, and so is $g(x)$.

2. First let's show that if the restriction of f to B and $E\sim B$ are measurable then f is measurable on E.

Notice that:

$$\{x \in E |\, f(x) > c\} = \{x \in B |\, f(x) > c\} \cup \{x \in (E\sim B) |\, f(x) > c\}.$$

Each set on the RHS is measurable so f is measurable.

Now let's show that if f is measurable on E then the restriction of f to B and $E\sim B$ are measurable.

$$\{x \in B |\, f(x) > c\} = \{x \in E |\, f(x) > c\} \cap B$$

$$\{x \in (E\sim B) |\, f(x) > c\} = \{x \in E |\, f(x) > c\} \cap (E\sim B)$$

In each case the RHS is measurable so the restriction of f to B and $E\sim B$ are measurable.

Thus f is measurable if and only if the restrictions of f to B and $E\sim B$ are.

Theorem: Let f and g be measurable function on E that are finite a.e. on E.

1. For any $a, b \in \mathbb{R}$, $af + bg$ is measurable on E.
2. fg is measurable on E.

Note: We need f, g to be finite a.e. because at points where $f(x) = \infty$ and $g(x) = -\infty$, for example, $f + g$ is not well defined.

Proof: If $a = 0$, then $af = 0$ where f is finite (i.e. a.e.), hence af is measurable. If $a \neq 0$ then:

$$\{x \in E |\, af(x) > c\} = \{x \in E |\, f(x) > \tfrac{c}{a}\}; \quad \text{if } a > 0$$

$$\{x \in E |\, af(x) > c\} = \{x \in E |\, f(x) < \tfrac{c}{a}\}; \quad \text{if } a < 0.$$

Thus f measurable implies that af is measurable.

Now we just need to show $f + g$ is measurable.

For each $x \in E$, if $f(x) + g(x) < c$, then $f(x) < c - g(x)$.

Since $\mathbb{Q}$ is dense in $\mathbb{R}$, there is a rational number, q, for which

$$f(x) < q < c - g(x) \quad \text{or} \quad f(x) < q \quad \text{and} \quad g(x) < c - q.$$

Hence:

$$\{x \in E |\, f(x) + g(x) < c\}$$
$$= \cup_{q \in \mathbb{Q}} [x \in E | g(x) < c - q\} \cap \{x \in E |\, f(x) < q\}].$$

$\{x \in E |\, f(x) + g(x) < c\}$ is measurable because it's a countable union of measurable sets.

Hence $f + g$ is measurable.

2 To prove fg is measurable, note that:

$$fg = \frac{1}{2}[(f + g)^2 - f^2 - g^2].$$

So we just need to show f^2 is measurable when f is measurable.

For $c \geq 0$:

$$\{x \in E | f^2(x) > c\} = \left\{x \in E \middle| f(x) > \sqrt{c}\right\} \cup \left\{x \in E \middle| f(x) < -\sqrt{c}\right\}.$$

Both sets on the RHS are measurable so the LHS is.

For $c < 0$:

$\{x \in E | f^2(x) > c\} = E$; where E is measurable.

Thus $f^2$is measurable.

Note: Although continuity and differentiability are preserved under the composition of functions, measurability is not. That is, there exist measurable functions f, g such that $f(g(x))$ is not measurable. However:

Prop. Let g be a measurable real valued function defined on E and f a continuous real valued function defined on all of $\mathbb{R}$. Then the composition $f(g(x))$ is a measurable function on E.

Proof: We show that given any open set O,

$$(f \circ g)^{-1}(O) = g^{-1}(f^{-1}(O)) \text{ is measurable.}$$

Since f is continuous, $f^{-1}(O)$ is open.

Since g is measurable, g^{-1} of an open set is measurable.

Hence $(f \circ g)^{-1}(O) = g^{-1}(f^{-1}(O))$ is measurable.

As a consequence of the previous proposition if f is measurable on E then so are $|f|$ and $|f|^p$ for $p > 0$.

Prop. For a finite family $\{f_k\}_{k=1}^n$ of measurable functions on E, $\max\{f_1(x), f_2(x), \dots, f_n(x)\}$ and $\min\{f_1(x), f_2(x), \dots, f_n(x)\}$ are measurable.

Proof: For any c:

$$\{x \in E | \max\{f_1(x), f_2(x), \dots, f_n(x)\} > c\} = \cup_{k=1}^n \{x \in E | f_k(x) > c\}$$

$$\{x \in E | \min\{f_1(x), f_2(x), \dots, f_n(x)\} < c\} = \cup_{k=1}^n \{x \in E | f_k(x) < c\}.$$

In each case the RHS is the finite union of measurable sets, hence

$\max\{f_1(x), f_2(x), \dots, f_n(x)\}$ and $\min\{f_1(x), f_2(x), \dots, f_n(x)\}$ are measurable.

When we discuss Lebesgue integration it will be useful to work with the functions:

$f^+(x) = \max\{f(x), 0\} \geq 0$

$f^-(x) = \max\{-f(x), 0\} \geq 0$

So if f is measurable on E then so are f^+ and f^-.

Also $f = f^+ - f^-$ on E.

Ex. Let $f: D \to \mathbb{R}$, where D is measurable. Show f is measurable if and only if $\{x \in D | f(x) > \alpha\}$ is measurable for each rational number α.

$\Longrightarrow$ If f is measurable then $\{x \in D | f(x) > c\}$ is measurable for any real number c, thus it's measurable for any rational number α.

$\Longleftarrow$ Assume $\{x \in D | f(x) > \alpha\}$ is measurable for each $\alpha \in \mathbb{Q}$.

Given any $c \in \mathbb{R}$, we can find a decreasing sequence $\{\alpha_k\} \to c$; $\alpha_k \in \mathbb{Q}$.

$\{x \in D | f(x) > c\} = \cup_{k=1}^{\infty}\{x \in D | f(x) > \alpha_k\}$;
So the LHS is measurable because it's the countable union of measurable sets.

Ex. Show that if $f, g: D \to \mathbb{R}$, D a measurable set and f, g measurable functions then $\{x \in D |\, f(x) > g(x)\}$ is measurable.

Since f, g are measurable functions, so is $f - g$.

Let $h(x) = f(x) - g(x)$.

$\{x \in D |\, f(x) > g(x)\} = \{x \in D |\, h(x) > 0\}$.

But h is measurable so $\{x \in D |\, f(x) > g(x)\}$ is measurable.