Approximating Lebesgue Measurable Sets

So far we know the following about measurable sets:

1. They form a σ-algebra (so closed under countable unions and complements hence also closed under countable intersections)
2. They contain all Borel sets (the smallest σ-algebra containing all open subsets of $\mathbb{R}$)
3. All sets of measure 0 are measurable.

Measurable sets contain the following **excision property**. If A is measurable of finite outer measure and $A \subseteq E$, then

$$m^*(E \sim A) = m^*(E) - m^*(A)$$

since:

$$m^*(E) = m^*(E \cap A) + m^*(E \cap A^c) = m^*(A) + m^*(E \sim A).$$

And since $m^*(A)$ is finite we have:

$$m^*(E \sim A) = m^*(E) - m^*(A).$$

Theorem: Let E be any set of real numbers. Then each of the following is equivalent to the measurability of E:

1. For each $\epsilon > 0$, there's an open set $O \supseteq E$ where $m^*(O \sim E) < \epsilon$
2. There is a G_δ set $G \supseteq E$ for which $m^*(G \sim E) = 0$
3. For each $\epsilon > 0$, there's an closed set $F \subseteq E$ where $m^*(E \sim F) < \epsilon$
4. There is an F_σ set $F \subseteq E$ where $m^*(E \sim F) = 0$.

Proof: Assume E is measurable and let $\epsilon > 0$ be given.

If $m^*(E) < \infty$ then by the definition of an outer measure there is a collection $\{I_k\}_{k=1}^{\infty}$ of open intervals which covers E and has

$$\sum_{k=1}^{\infty} l(I_k) < m^*(E) + \epsilon.$$

Let $O = \bigcup_{k=1}^{\infty} I_k$, then O is an open set containing E. In addition:

$$m^*(O) \le \sum_{k=1}^{\infty} l(I_k) < m^*(E) + \epsilon,$$

So $$m^*(O) - m^*(E) < \epsilon .$$

Since E is measurable and has finite measure, by the excision property:

$$m^*(O{\sim}E) = m^*(O) - m^*(E) < \epsilon.$$

If $m^*(E) = \infty$, let $E_k = E \cap [k, k+1]$ and $E = \bigcup_{k\in\mathbb{Z}} E_k$, a countable union measurable sets, each with finite measure.

By the first part we know there is an open set $O_k \supseteq E_k$ with

$$m^*(O_k{\sim}E_k) < \frac{\epsilon}{2^{|k|+2}}.$$

Now let $O = \bigcup_{k\in\mathbb{Z}} O_k$.

O is open, $O \supseteq E$, and $O{\sim}E = \bigcup_{k\in\mathbb{Z}} O_k \sim E \subseteq \bigcup_{k\in\mathbb{Z}}(O_k{\sim}E_k)$.

(Draw a picture with 2 pairs of sets, O_1, O_2, and E_1, E_2 to see this is true.)

Therefore:

$$m^*(O{\sim}E) \le \sum_{k\in\mathbb{Z}} m^*(O_k{\sim}E_k) < \sum_{k\in\mathbb{Z}} \frac{\epsilon}{2^{|k|+2}} < \epsilon$$

$1 \Rightarrow 2$:

For each $\epsilon > 0$, there's an open set $O \supseteq E$ where $m^*(O{\sim}E) < \epsilon$.

For each $n > 0$, choose $O_n \supseteq E$ an open set with $m^*(O_n{\sim}E) < \frac{1}{n}$.

Let $G = \bigcap_{n=1}^{\infty} O_n$.

G is a G_δ set and $G \supseteq E$.

In addition, for each n, $G{\sim}E \subseteq O_n{\sim}E$. Thus

$m^*(G{\sim}E) \leq m^*(O_n{\sim}E) < \frac{1}{n}$, for all n.

Hence $m^*(G{\sim}E) = 0$.

Now let's show that $2 \Rightarrow E$ is measurable.

Since there is a G_δ set $G \supseteq E$ for which $m^*(G{\sim}E) = 0$,

$G{\sim}E$ has measure 0 and is hence measurable.

G is a G_δ set, hence it's measurable. Thus E is measurable because

$$E = G \cap (G{\sim}E)^c.$$

3 and 4 follow from the fact that a set is measurable if and only if its complement is measurable, is open if and only if its complement is closed, is F_σ if and only if its complement if G_δ, and

$E{\sim} \bigcup_{k=1}^{\infty} V_k = \bigcap_{k=1}^{\infty}(E{\sim}V_k)$, $\quad$ $E{\sim} \bigcap_{k=1}^{\infty} V_k = \bigcup_{k=1}^{\infty}(E{\sim}V_k)$.

(DeMorgan Identities).

Ex. show with an example that It is not true that

a. if E is measurable then there exists a closed set F such that if $E \subseteq F$ and $m^*(F \sim E) < \epsilon$

b. if E is measurable then there exists an open set O such that $O \subseteq E$ and $m^*(E \sim O) < \epsilon$.

a. Let $E = \mathbb{Q} \cap [0,1]$.

Then any closed set $F \supseteq E$ must have $F \supseteq [0,1]$, since F must contain all limit points of E.

Hence $m^*(F) \geq 1$, but $m^*(E) = 0$.

Thus if $\epsilon = \frac{1}{2}$, for example, then $m^*(F \sim E) \nless \epsilon$.

b. Let $E = [0,1] \sim (\mathbb{Q} \cap [0,1])$ =set of irrational numbers between 0 and 1.

But the only open set E contains is the empty set, ϕ.

Thus $m^*(E) = 1$ and $m^*(\phi) = 0$.

Hence if $\epsilon = \frac{1}{2}$, for example, then $m^*(E \sim \phi) \nless \epsilon$.

Ex. Use as a definition of a measurable set that E is measurable if there exists a G_δ set $G \supseteq E$ for which $m^*(G{\sim}E) = 0$ and prove that the union of two measurable sets is measurable.

Let D and E be measurable sets.

Thus there exists G_δ sets $G \supseteq D$ and $H \supseteq E$ with $m^*(G{\sim}D) = 0$ and $m^*(H{\sim}E) = 0$.

We must show there exists a G_δ set $K \supseteq D \cup E$ with $m^*(K{\sim}(D \cup E)) = 0.$

Let's show $K = G \cup H$ works.

K is a G_δ set because it's the union of G_δ sets.

$G{\sim}D = G \cap D^c,$ and $H{\sim}E = H \cap E^c.$

Notice that:

$(G \cup H){\sim}(D \cup E) = (G \cup H) \cap (D \cup E)^c \subseteq (G \cap D^c) \cup (H \cap E^c)$

(draw a picture to see that this is true)

Thus: $m^*((G \cup H){\sim}(D \cup E)) \leq m^*((G \cap D^c) \cup (H \cap E^c))$

$$\leq m^*\big((G \cap D^c)\big) + m^*\big((H \cap E^c)\big)$$

$$= m^*(G{\sim}D) + m^*(H{\sim}E) = 0 + 0 = 0.$$

Thus $D \cup E$ is measurable.

Theorem: Let E be a measurable set of finite outer measure. Then for each $\epsilon > 0$, there is a finite disjoint collection of open intervals $\{I_k\}_{k=1}^{n}$ for which if $O = \bigcup_{k=1}^{n} I_k$, then

$$m^*(E \sim O) + m^*(O \sim E) < \epsilon.$$

Proof: We know that if E is measurable then there is an open set U such that $E \subseteq U$ and $m^*(U \sim E) < \frac{\epsilon}{2}$.

Since U is open it's measurable and has finite measure because E does.

Every open set of real numbers is the union of a countable collection of disjoint open intervals $\{I_j\}_{j=1}^{\infty}$, thus $U = \bigcup_{j=1}^{\infty} I_j$.

Now we know that for all n:

$$\sum_{j=1}^{n} l(I_j) = m^*(\bigcup_{j=1}^{n} I_j) \leq m^*(U) < \infty.$$

Thus $\sum_{j=1}^{\infty} l(I_j) < \infty$.

Choose an n such that $\sum_{j=n+1}^{\infty} l(I_j) < \frac{\epsilon}{2}$.

Define $O = \bigcup_{j=1}^{n} I_j$.

Since $O \sim E \subseteq U \sim E$ we have:

$$m^*(O \sim E) \leq m^*(U \sim E) < \frac{\epsilon}{2}.$$

However, since $E \subseteq U$

$$E \sim O \subseteq U \sim O = \bigcup_{j=n+1}^{\infty} I_j.$$

So: $m^*(E \sim O) \leq \sum_{j=n+1}^{\infty} l(I_j) < \frac{\epsilon}{2}$.

Thus we have: $m^*(E \sim O) + m^*(O \sim E) < \epsilon$.