Bounded Linear Functionals on L^p Spaces

Def. A **linear functional** on a linear space X is a real valued function T on X such that for $g, h \in X$ and $\alpha, \beta \in \mathbb{R}$

$$T(\alpha g + \beta h) = \alpha T(g) + \beta T(h).$$

Notice that if T, S are linear functionals on X, so is $aT + bS,\ \ a, b \in \mathbb{R}$.

Since the functional $T: X \to \mathbb{R}$ defined by $T(g) = 0$ for all $g \in X$ is linear, the set of linear functionals on X is itself a linear space.

Ex. Let E be a measurable set, $1 \leq p < \infty$ and q the conjugate of p. If $g \in L^q(E)$ then define:

$$T: L^p(E) \to \mathbb{R}$$

by $$T(f) = \int_E fg; \quad f \in L^p(E).$$

By the Holder inequality $fg \in L^1(E)$.

T is linear because integration over E is linear. Notice also that:

$$|T(f)| = \left| \int_E fg \right| \leq \int_E |fg| \leq \|f\|_p \|g\|_q$$

For all $f \in L^p(E)$.

Def. For a normed linear space X, a linear functional is said to be ***bounded*** if there is an $M \geq 0$ such that:

$$|T(f)| \leq M\|f\| \quad \text{for all } f \in X.$$

The infimum of all such M is called the norm of T, denoted $\|T\|_*$.

Ex. $T: L^p(E) \to \mathbb{R}$ by $T(f) = \int_E \ fg; \quad f \in L^p(E)$, with a fixed $g \in L^q(E)$, is a bounded linear functional since

$$|T(f)| \leq \int_E \ |fg| \leq \|f\|_p \|g\|_q, \qquad (M = \|g\|_q).$$

Let T be a bounded linear functional on X, and $M = \|T\|_*$. Then for any $f, h \in X$:

$$|T(f) - T(h)| = |T(f-h)| \leq \|T\|_* \|f - h\|.$$

Thus if $f_n \to f$ in X then :

$$|T(f_n) - T(f)| \leq \|T\|_* \|f_n - f\|.$$

So if $\lim_{n\to\infty} \|f_n - f\| = 0$ then $\lim_{n\to\infty} |T(f_n) - T(f)| = 0.$

That is, if T is a bounded linear functional on X and $f_n \to f$ in X then the sequence of real numbers $\{T(f_n)\}$ converges to $T(f)$ in $\mathbb{R}$, ie T is a continuous map from X to $\mathbb{R}$.

Because T is linear

$$\|T\|_* = \inf\{M | \ |T(f)| \leq M\|f\|, for\ all\ f \in X\}$$
$$= \sup\{T(f) | \ f \in X, \ \|f\| \leq 1\}$$

Since if $f \neq 0$ then $|T\left(\frac{f}{\|f\|}\right)| \leq M \iff |T(f)| \leq M\|f\|.$

Prop. Let X be a normed linear space. Then the collection of bounded linear functionals on X is a linear space on which $\| \ \ \|_*$ is a norm. This normed linear space is called the **dual space of X** and is denoted X^*.

Prop. Let E be a measurable set, $1 \le p < \infty$, q the conjugate of p, and $g \in L^q(E)$. Define T on $L^p(E)$ by $T(f) = \int_E gf$ for all $f \in L^p(E)$. Then T is a bounded linear functional on $L^p(E)$ and $\|T\|_* = \|g\|_q$.

Proof: We already saw that T is a bounded linear functional on $L^p(E)$ and $\|T\|_* \le \|g\|_q$.

Assume $p > 1$.

To show $\|T\|_* = \|g\|_q$ let's find a function f such that:

$$|T(f)| = \|g\|_q \|f\|_p.$$

Let $f = sgn(g)|g|^{\frac{q}{p}}$.

$$T(f) = \int_E \left(sgn(g)\right)|g|^{\frac{q}{p}} g = \int_E |g|^{\frac{q}{p}}|g| = \int_E |g|^{(1+\frac{q}{p})} = \int_E |g|^q.$$

Now let's show $T(f) = \int_E |g|^q = \|g\|_q \|f\|_p$.

$$\|f\|_p = [\int_E (|g|^{\frac{q}{p}})^p]^{\frac{1}{p}} = [\int_E |g|^q]^{\frac{1}{p}}$$

$$\|g\|_q = [\int_E |g|^q]^{\frac{1}{q}}$$

$$\|g\|_q \|f\|_p = [\int_E |g|^q]^{\frac{1}{q}} [\int_E |g|^q]^{\frac{1}{p}}$$

$$= [\int_E |g|^q]^{\left(\frac{1}{p}+\frac{1}{q}\right)} = \int_E |g|^q = T(f).$$

So $\|T\|_* = \|g\|_q$.

If $p = 1$ we argue by contradiction.

If $\|g\|_\infty > \|T\|_*$ then there is A with $m(A) > 0$, where $|g(x)| > \|T\|_*$.

For $x \in A$ let $f = \frac{1}{m(A)}(sgn(g))\chi_A$.

Then $\|f\|_1 = \int_E \left|\frac{1}{m(A)}(sgn(g))\chi_A\right| = 1$.

But we have:

$$T(f) = \int_E \frac{1}{m(A)}(sgn(g))\chi_A g = \int_E \frac{1}{m(A)}\chi_A|g| = \int_A \frac{1}{m(A)}|g| > \|T\|_*$$

Which is a contradiction, so $\|g\|_\infty = \|T\|_*$.

Our goal is to show that every bounded linear functional on $L^p(E)$ looks like $T(f) = \int_E fg$ for some $g \in L^q(E)$.

Prop. Let T and S be bounded linear functionals on a normed linear space X. If $T = S$ on a dense subset X_0 of X then $T = S$ on X.

Proof: Let $g \in X$.

Since X_0 is dense in X there is a sequence $\{g_n\}$ in X_0 such that $g_n \to g$ in X. Since S and T are bounded linear functionals:

$$T(g_n) \to T(g) \quad \text{and} \quad S(g_n) \to S(g).$$

But since $S(g_n) = T(g_n)$ for all n, $T(g) = S(g)$.

So $T = S$ on X.

Lemma: Let E be a measurable set and $1 \leq p < \infty$. Suppose g is integrable over E and there is an $M \geq 0$ for which

$$\int_E gf \leq M\|f\|_p \text{ for every simple function } f \in L^p(E).$$

Then $g \in L^q(E)$, where q is the conjugate of p and $\|g\|_q \leq M$.

Outline of Proof: For $p > 1$.

The Simple Approximation Theorem says there exists $\varphi_n \to |g|$, φ_n simple and $0 \leq \varphi_n \leq |g|$.

Thus ${\varphi_n}^q \to |g|^q$.

So by Fatou's lemma: $\int_E |g|^q \leq liminf \int_E {\varphi_n}^q$.

Now show $\int_E {\varphi_n}^q \leq M^q$, and thus $g \in L^q(E)$ and $\|g\|_q \leq M$.

If $p = 1$, show $M = esssup(g)$.

Assume $M > esssup(g)$ and get a contradiction.

Let $E_\epsilon = \{x \in E| \, |g(x) > M + \epsilon\}$ with $m(E_\epsilon) > 0$.

Let $f = \chi_{E_\epsilon}(sgn(g))$ then

$|\int_E fg| = \int_{E_\epsilon} |g| \geq (M+\epsilon)(m(E_\epsilon)) = (M+\epsilon)\|f\|_1$.

Which is a contradiction.

Thus $M = esssup(g)$.

Theorem: Let $[a,b]$ be a closed, bounded interval and $1 \leq p < \infty$. Suppose T is a bounded linear functional on $L^p[a,b]$. Then there is a function $g \in L^q[a,b]$, where q is the conjugate of p, such that

$$T(f) = \int_{[a,b]} fg \quad \text{for all } f \in L^p[a,b].$$

Outline of Proof: Let $\Phi(x) = T\big(\chi_{[a,x)}\big)$.

Show that $\Phi(x)$ is absolutely continuous and hence

$$\Phi(x) = \Phi(a) + \int_a^x g, \quad \text{where } g = \Phi' \text{ and } x \in [a,b].$$

Now show $T(f) = \int_{[a,b]} fg$ when f is a step function.

Now show $T(f) = \int_{[a,b]} fg$ when f is a simple function, by taking a sequence of step functions φ_n which converge to f.

The previous lemma now shows $g \in L^q[a,b]$.

Simple functions are dense in $L^p[a,b] \Longrightarrow$ result for $f \in L^p[a,b]$.

The Riesz Representation Theorem for the dual of $L^p(E)$.

Let E be measurable and $1 \leq p < \infty$, and q the conjugate of p. For each $g \in L^q(E)$, define the bounded linear functional R_g on $L^p(E)$ by

$$R_g(f) = \int_E gf \quad \text{for all } f \in L^p(E).$$

Then for each bounded linear functional T on $L^p(E)$, there is a unique $g \in L^q(E)$ for which $R_g = T$ and $\|T\|_* = \|g\|_q$.

Outline of Proof: First prove uniqueness, i.e., if $R_{g_1} = R_{g_2}$ then $g_1 = g_2$.

By linearity $R_{g_1} - R_{g_2} = R_{g_1 - g_2}$, so $R_{g_1} = R_{g_2} \Longrightarrow R_{g_1 - g_2} = 0$.

But then $\|g_{1-}g_2\|_q = 0$, thus $g_1 = g_2$ a.e.

The previous theorem gives the result for $E = [-n, n]$, i.e.

$R_g(f) = T_n(f) = \int_{-n}^{n} g_n f$; $\quad g_n \in L^q[-n, n]$, $\|T_n\|_* = \|g_n\|_q$.

Consider the sequence of $\{|g_n|^q\}$ converging to $|g|^q$.

Applying Fatou's lemma:

$$\int_{\mathbb{R}} |g|^q \leq liminf \int_{\mathbb{R}} |g_n|^q \leq \|T_n\|_*{}^q \leq \|T\|_*{}^q$$

So $g \in L^q(\mathbb{R})$.

Mow get the result for $f \in L^p(\mathbb{R})$, where f vanishes outside a bounded set which is dense in $L^p(\mathbb{R})$. The result for $L^p(\mathbb{R})$ follows from denseness.

To get the result for $E \subseteq \mathbb{R}$, define $\overline{T}(f) = T(f|_E)$.

$\overline{T} = R_{\overline{g}}$ for some $\overline{g} \in L^q(\mathbb{R})$.

Now define $g = \overline{g}|_E$ then $T = R_g$.