Approximation of L^p Functions

We have already seen that if $f \in L^1(\mathbb{R})$ then given any $\epsilon > 0$ there is a simple function η, a step function s, and a continuous function g, all with finite support such that:

$$\int_{\mathbb{R}} |f - \eta| < \epsilon, \quad \int_{\mathbb{R}} |f - s| < \epsilon, \quad \text{and} \quad \int_{\mathbb{R}} |f - g| < \epsilon.$$

We will now extend these results to $f \epsilon L^p(E)$.

Def. Let X be a normed linear space. Given two subsets F and G of X with $F \subseteq G$, we say F is **dense** in G if for each $g \in G$ and $\epsilon > 0$ there is an element $f \in F$ where $\|f - g\| < \epsilon$.

Notice that F is dense in G if and only if for each $g \in G$ there is a sequence $\{f_n\}$ in F such that $\lim_{n\to\infty} f_n = g$ in X.

In addition for $F \subseteq G \subseteq H \subseteq X$ if F is dense in G and G is dense in H then F is dense in H.

Ex. The rational numbers are dense in $\mathbb{R}$.

Ex. Simple functions, step functions, and continuous functions, all with finite support are dense in $L^1(\mathbb{R})$ (and $L^1(E)$).

Prop. Let E be a measurable set and $1 \leq p \leq \infty$. Then the simple functions in $L^p(E)$ are dense in $L^p(E)$.

Proof: Let $f \in L^p(E)$.

First let $p = \infty$.

By definition, there is a subset of measure 0, E_0, such that f is bounded on $E{\sim}E_0$.

From the simple approximation lemma, there is a sequence of simple functions $\{\varphi_n\}$ on $E{\sim}E_0$ such that $|\varphi_n| \leq |f|$ for all n on $E{\sim}E_0$ with $\varphi_n \to f$ uniformly on $E{\sim}E_0$.

Since $f \in L^\infty(E)$, it is bounded a.e and since $\varphi_n \to f$ uniformly on $E{\sim}E_0$, it converges in $L^\infty(E)$.

Thus simple functions are dense in $L^\infty(E)$.

Now assume $1 \leq p < \infty$.

By the simple approximation theorem, there is a sequence of simple functions $\varphi_n \to f$ pointwise on E with $|\varphi_n| \leq |f|$ for all n.

By the integral comparison test $\varphi_n \in L^p(E)$ for all n since

$$|\varphi_n|^p \leq |f|^p \text{ thus } \int_E |\varphi_n|^p \leq \int_E |f|^p < \infty.$$

Now notice that:

$$|\varphi_n - f|^p \leq 2^p(|\varphi_n|^p + |f|^p) \leq 2^{p+1}|f|^p \quad \text{on } E.$$

Since $|f|^p$ is integrable over E, Lebesgue dominated convergence says:

$$\lim_{n\to\infty} \int_E \ |\varphi_n - f|^p = \int_E \lim_{n\to\infty} |\varphi_n - f|^p = 0.$$

So $\varphi_n \to f$ in $L^p(E)$ and simple functions are dense in $L^p(E)$.

Prop. Let $[a, b]$ be a closed, bounded interval and $1 \leq p < \infty$. Then the set of step functions is dense in $L^p[a, b]$.

Proof: By the previous proposition we only need to show that step functions are dense in simple functions on $[a, b]$.

Every simple function is: $\varphi = \sum_{i=1}^{n} a_i \chi_{A_i}$; $A_i \subseteq [a, b]$ measurable.

Thus we only need to show we can approximate a characteristic function $\varphi = \chi_A$, where A is measurable.

Let $\epsilon > 0$.

We need a step function s on $[a, b]$ such that $\|\varphi - s\|_p < \epsilon$.

Since A is a set of finite measure, for any $\epsilon > 0$ there exists a finite, disjoint collection of open intervals $\{I_k\}_{k=1}^{n}$ for which if $O = \bigcup_{k=1}^{n} I_k$, then $m(A{\sim}O) + m(O{\sim}A) < \epsilon^p$.

Now χ_O is a step function and

$$\|\chi_A - \chi_O\|_p = [m(A{\sim}O) + m(O{\sim}A)]^{\frac{1}{p}} < (\epsilon^p)^{\frac{1}{p}} = \epsilon$$

Thus step functions are dense in $L^p[a, b]$.

Def. A normed linear space X is said to be **separable** if there is a countable subset that is dense in X.

Ex. The real numbers are separable because $\mathbb{Q}$ is countable and dense in $\mathbb{R}$.

Theorem: Let E be a measurable set and $1 \leq p < \infty$. Then $L^p(E)$ is separable.

Proof: Let $[a, b]$ be a closed bounded interval.

Let $S[a, b] = \{step\ functions\ s\ on\ [a, b]\}$.

Let $S_{\mathbb{Q}}[a, b] \subseteq S[a, b]$ be step functions that take on rational values and for which there is a partition $P = \{x_0, x_1, \dots, x_n\}$ of $[a, b]$ with s constant on (x_{k-1}, x_k), $1 \leq k \leq n$, and x_k rational for $1 \leq k \leq n - 1$.

Since $\mathbb{Q}$ is dense in $\mathbb{R}$, $S_{\mathbb{Q}}[a, b]$ is dense in $S[a, b] \subseteq L^p[a, b]$.

In addition, $S_{\mathbb{Q}}[a, b]$ is countable.

Since $S[a, b]$ is dense in $L^p[a, b]$, $S_{\mathbb{Q}}[a, b]$ is dense in $L^p[a, b]$.

Now define F_n to be functions on $\mathbb{R}$ that vanish outside $[-n, n]$ and whose restriction to $[-n, n]$ belong to $S_{\mathbb{Q}}[-n, n]$.

Let $F = \bigcup_{n \in \mathbb{Z}^+} F_n$, which is again countable.

By the monotone convergence theorem:

$$\lim_{n\to\infty} \int_{-n}^{n} |f|^p = \int_{\mathbb{R}} |f|^p \text{ for all } f \in L^p(\mathbb{R}).$$

We just showed that there is a step function in $S_{\mathbb{Q}}[-n,n]$ that approximates $f \in L^p[-n,n]$ and thus $F = \bigcup_{n\in\mathbb{Z}^+} F_n$ contains a function that approximates $f \in L^p(\mathbb{R})$.

Thus $L^p(\mathbb{R})$ is separable.

If E is a general measurable set, then the collection of restrictions to E of F is a countable dense subset of $L^p(E)$, and $L^p(E)$ is separable.

In fact, we just showed that step functions with finite support are dense in $L^p(E)$, where E is measurable and $1 \leq p < \infty$.

Ex. Step functions with finite support are not dense in $L^\infty(E)$ if $m(E) = \infty$. For example, let's take $L^\infty(\mathbb{R})$.

$f(x) = 1$ is in $L^\infty(\mathbb{R})$, but if φ is any step function with finite support then $\|f - \varphi\|_\infty \geq 1,$ Thus step functions with finite support are not dense in $L^\infty(\mathbb{R})$.

Let $C_c(E) = \{Continuous\ functions\ on\ E\ with\ finite\ support\}$.

Theorem: Let E be measurable and $1 \leq p < \infty$, then $C_c(E)$ is dense in $L^p(E)$.

Proof: Since the set of step functions with finite support is dense in $L^p(E)$, one just needs to show that $C_c(E)$ is dense in the set of step functions with finite support.

See the theorem where we showed that $C_c(E)$ is dense in $L^1(\mathbb{R})$.