Completeness of L^p: The Riesz-Fischer Theorem

Def. A sequence $\{f_n\}$ in a normed linear space X is said to **converge to $\boldsymbol{f} \in \boldsymbol{X}$** if $\lim_{n\to\infty} \|f - f_n\| = 0$. In that case we write $f_n \to f$ or $\lim_{n\to\infty} f_n = f$ in X.

Ex. Let $X = C[a,b]$, with $\|f\| = \max_{x\in[a,b]} |f(x)|$.

In this case $f_n \to f$ means $\lim_{n\to\infty} \max_{x\in[a,b]} |f(x) - f_n(x)| = 0$ or

for all $\epsilon > 0$ there exists N such that if $n \geq N$ then

$$\|f - f_n\| = \max_{x\in[a,b]} |f(x) - f_n(x)| < \epsilon.$$

This is precisely the definition of uniform convergence on $[a,b]$.

Ex. Let $X = L^\infty[a,b]$, with $\|f\| =$Essential Supremum(f).

In this case $f_n \to f$ means $\lim_{n\to\infty} EssSup(f - f_n) = 0$ or

for all $\epsilon > 0$ there exists N such that if $n \geq N$ then

$$\|f - f_n\| = Essential\ Sup|f - f_n| < \epsilon.$$

This is the same as saying that $f_n \to f$ in $L^\infty[a,b]$ if an only if $f_n \to f$ uniformly on the complement of a set of measure 0 in $[a,b]$.

Ex. Let $X = L^p(E)$ with $\|f\|_p = (\int_E |f|^p)^{\frac{1}{p}}$.

$f_n \to f$ in $L^p(E)$ if and only if $0 = \lim_{n\to\infty} \|f - f_n\| = \lim_{n\to\infty} (\int_E |f - f_n|^p)^{\frac{1}{p}}$

Or for all $\epsilon > 0$ there exists N such that if $n \geq N$ then $(\int_E |f - f_n|^p)^{\frac{1}{p}} < \epsilon$.

Def. A sequence $\{f_n\}$ is a normed linear space X is said to be **Cauchy** in X if for each $\epsilon > 0$ there is an N such that if $n, m \geq N$ then $\|f_n - f_m\| < \epsilon$.

Def. A normed linear space X is said to be **complete** if every Cauchy sequence in X converges to a point (function) $f \in X$. A complete normed linear space is called a **Banach** space.

There can be more than one way to define a norm on a linear space X. For example, if $X = C[0,1]$ we could define:

$$\|f\| = \max_{x\in[0,1]} |f(x)| \quad \text{or} \quad \|f\| = \int_0^1 |f|.$$

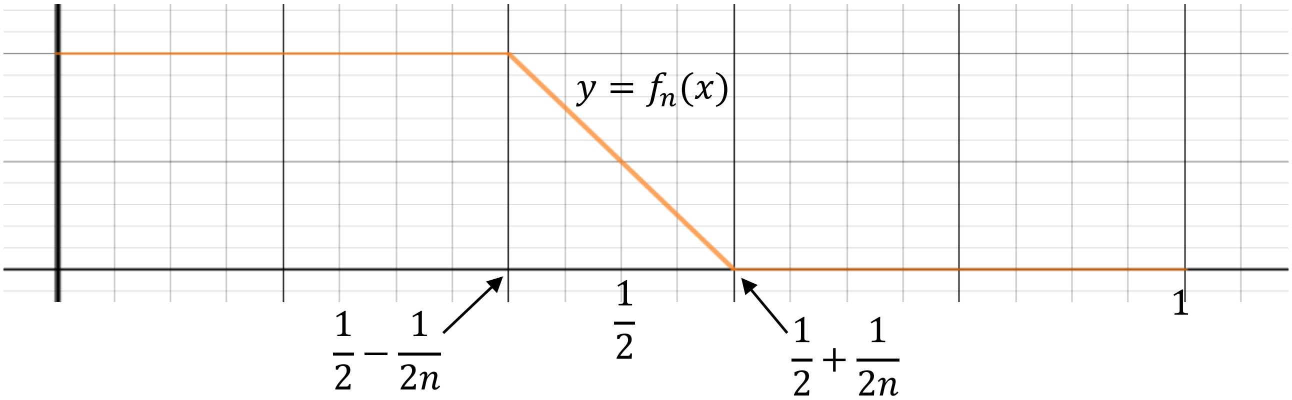
Whether a normed linear space is complete depends on which norm you choose.

Ex. $C[0,1]$ is a Banach space with the norm $\|f\| = \max_{x\in[0,1]} |f(x)|$, but is not a Banach space with the norm $\|f\| = \int_0^1 |f|$.

One learns in an undergraduate analysis course that a uniformly convergent sequence of continuous functions converges to a continuous function. This says $C[0,1]$ is complete with $\|f\| = \max_{x\in[0,1]} |f(x)|$.

However, if we let

$$f_n(x) = \begin{cases} 1 & \text{if } 0 \leq x \leq \frac{1}{2} - \frac{1}{2n} \\ -nx + \frac{n+1}{2} & \text{if } \frac{1}{2} - \frac{1}{2n} < x < \frac{1}{2} + \frac{1}{2n} \\ 0 & \text{if } \frac{1}{2} + \frac{1}{2n} \leq x \leq 1 \end{cases}$$



Then $\{f_n\}$ is a Cauchy sequence with respect to $\|f\| = \int_0^1 |f|$, but it does not converge to an element of $C[0,1]$. Hence $C[0,1]$ is not complete with respect to the norm $\|f\| = \int_0^1 |f|$.

We will see that all of the $L^p(E)$ spaces, $1 \le p \le \infty$ are Banach spaces with respect to their standard norms.

Prop. Let X be a normed linear space. Then every convergent sequence in X is a Cauchy sequence in X. Moreover, a Cauchy sequence in X converges if it has a convergent subsequence.

Proof: Suppose $f_n \to f$ in X.

Then by the triangle inequality, for all m, n:

$$\|f_n - f_m\| = \|(f_n - f) + (f - f_m)\| \le \|f_n - f\| + \|f - f_m\|.$$

Since $f_n \to f$, given any $\epsilon > 0$ there is a N such that if $n \ge N$ then

$$\|f - f_n\| < \frac{\epsilon}{2}.$$

Thus if $m, n \geq N$ then

$$\|f_n - f_m\| \leq \|f_n - f\| + \|f - f_m\| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon .$$

Hence $\{f_n\}$ is a Cauchy sequence.

Now let $\{f_n\}$ be a Cauchy sequence in X that has a convergent subsequence $\{f_{n_k}\}$.

Let $\epsilon > 0$.

$\{f_n\}$ is Cauchy so there is a N' such that $m, n \geq N' \Longrightarrow \|f_n - f_m\| < \frac{\epsilon}{2}$.

Since $\{f_{n_k}\}$ converges to $f \in X$ we can choose a k such that $n_k \geq N''$ then:

$$\left\|f_{n_k} - f\right\| < \frac{\epsilon}{2} .$$

Choose $N = \max{(N', N'')}$ then

$$\|f - f_n\| = \left\|\left(f - f_{n_k}\right) + (f_{n_k} - f_n)\right\|$$

$$\leq \left\|\left(f - f_{n_k}\right)\right\| + \left\|\left(f_{n_k} - f_n\right)\right\| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

Thus $f_n \to f$ in X.

Def. Let X be a normed linear space. A sequence $\{f_n\}$ is said to be **rapidly Cauchy** if there is a convergent series of positive numbers $\sum_{k=1}^{\infty} \epsilon_k$ such that

$$\|f_{k+1} - f_k\| \leq \epsilon_k^2 \text{ for all } k.$$

Ex. $\{\frac{1}{n^2}\}$ is rapidly Cauchy in $\mathbb{R}$, but $\{\frac{1}{n}\}$ is not rapidly Cauchy in $\mathbb{R}$.

For the sequence $\left\{\frac{1}{n^2}\right\}$:

$|\frac{1}{(k+1)^2} - \frac{1}{k^2}| = \frac{2k+1}{k^2(k+1)^2}$.

To be rapidly Cauchy we need: $\sum_{k=1}^{\infty} \sqrt{\frac{2k+1}{k^2(k+1)^2}} = \sum_{k=1}^{\infty} \frac{\sqrt{2k+1}}{k(k+1)}$ to converge.

This does converge through the limit comparison test with the series $\sum_{k=1}^{\infty} \frac{1}{k^{\frac{3}{2}}} < \infty$.

So $\{\frac{1}{n^2}\}$ is rapidly Cauchy in $\mathbb{R}$.

For the sequence $\left\{\frac{1}{n}\right\}$:

$|\frac{1}{k+1} - \frac{1}{k}| = \frac{1}{k(k+1)}$.

To be rapidly Cauchy we need: $\sum_{k=1}^{\infty} \frac{1}{\sqrt{k(k+1)}}$ to converge.

But this series diverges by the limit comparison test with $\sum_{k=1}^{\infty} \frac{1}{k} = \infty$.

Thus $\{\frac{1}{n}\}$ is not rapidly Cauchy in $\mathbb{R}$.

Notice that if $\{f_n\}$ is a sequence in X and we have a sequence of nonnegative numbers $\{a_k\}$ with

$$\|f_{k+1} - f_k\| \leq a_k \text{ for all } k \text{ then}$$

$$f_{n+k} - f_k = \sum_{j=n}^{n+k-1}(f_{j+1} - f_j) \text{ for all } n, k.$$

So

$$\|f_{n+k} - f_k\| = \left\|\sum_{j=n}^{n+k-1}(f_{j+1} - f_j)\right\| \leq \sum_{j=n}^{n+k-1}\left\|f_{j+1} - f_j\right\|$$
$$\leq \sum_{j=1}^{\infty} a_j \quad \text{for all } n, k.$$

Prop. Let X be a normed linear space. Then every rapidly Cauchy sequence in X is Cauchy. Furthermore, every Cauchy sequence has a rapidly Cauchy subsequence.

Proof: Let $\{f_n\}$ be rapidly Cauchy and $\sum_{k=1}^{\infty} \epsilon_k < \infty$ for which

$$\|f_{k+1} - f_k\| \leq \epsilon_k^2 \text{ for all } k.$$

Thus:

$$\|f_{n+k} - f_n\| = \|(f_{n+1} - f_n) + (f_{n+2} - f_{n+1}) + \cdots + (f_{n+k} - f_{n+k-1})\|$$
$$\leq \|f_{n+1} - f_n\| + \cdots + \|f_{n+k} - f_{n+k-1}\| \leq \sum_{j=n}^{\infty} \epsilon_j^2 \text{ for all } n, k.$$

Since $\sum_{k=1}^{\infty} \epsilon_k$ converges, $\sum_{k=1}^{\infty} {\epsilon_k}^2$ converges (by the comparison test).

Thus given $\epsilon > 0$ there exists an N such that $n \geq N$ implies

$$\|f_{n+k} - f_n\| \leq \left|\sum_{j=n}^{\infty} \epsilon_j^2\right| < \epsilon.$$

Thus $\{f_n\}$ is Cauchy.

Now assume $\{f_n\}$ is Cauchy.

We can always find an increasing sequence of $\{n_k\}$ such that

$$\left\|f_{n_{k+1}} - f_{n_k}\right\| < (\tfrac{1}{2})^k \quad \text{for all } k.$$

Thus $\{f_{n_k}\}$ is rapidly Cauchy because $\sum_{k=1}^{\infty}(\frac{1}{\sqrt{2}})^k$ converges because it's a geometric series with $r < 1$.

Theorem: Let E be a measurable set and $1 \le p \le \infty$. Then every rapidly Cauchy sequence in $L^p(E)$ converges both with respect to the $L^p(E)$ norm and pointwise a.e. on E to a function in $L^p(E)$.

Proof at end of this section.

The Riesz-Fischer Theorem: Let E be a measurable set and $1 \le p \le \infty$. Then $L^p(E)$ is a Banach space. Moreover if $f_n \to f$ in $L^p(E)$, a subsequence of $\{f_n\}$ converges pointwise a.e. on E to f.

Proof: Let $\{f_n\}$ be a Cauchy sequence in $L^p(E)$. Thus there is a subsequence $\{f_{n_k}\}$ that is rapidly Cauchy.

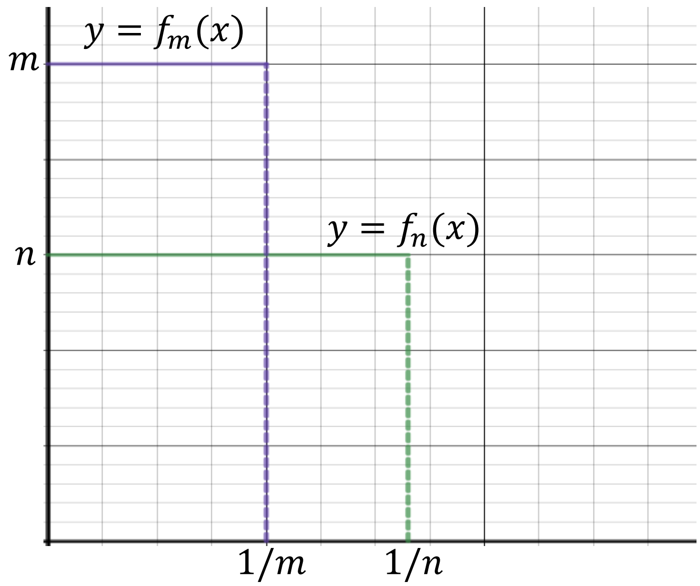
The previous theorem says that $f_{n_k} \to f$ in $L^p(E)$ and converges to f a.e. on E. Since a Cauchy sequence converges if it has a convergent subsequence, $f_n \to f$ in $L^p(E)$.

Ex. Pointwise convergence does not guarantee convergence in $L^p(E)$.

Let $f_n(x) = n$ if $0 < x < \frac{1}{n}$

$= 0$ if $\frac{1}{n} \leq x \leq 1.$

Then $f_n \in L^p(0,1)$ for all n and $f_n(x) \to f(x) = 0$ pointwise on $(0,1)$ but $\{f_n\}$ is not a Cauchy sequence in $L^p(0,1)$ and hence does not converge in $L^p(0,1)$.



For example in $L^1(0,1)$:

$$\|f_n - f_m\|_1 = \int_0^{\frac{1}{m}}(m-n) + \int_{\frac{1}{m}}^{\frac{1}{n}} n = 2 - \frac{2n}{m}; \text{ for } m > n$$

Which doesn't go to 0 as $n, m \to \infty$.

Ex. Convergence in $L^p(E)$, $1 \leq p < \infty$, does not guarantee pointwise convergence a.e. on E.

Let $f_1 = \chi_{[0,1]}$, $\quad f_2 = \chi_{[0,\frac{1}{2}]}$, $\quad f_3 = \chi_{[\frac{1}{2},1]}$, $\quad f_4 = \chi_{[0,\frac{1}{4}]}$,

$f_5 = \chi_{[\frac{1}{4},\frac{1}{2}]}$, $\quad f_6 = \chi_{[\frac{1}{2},\frac{3}{4}]}$, $\quad f_7 = \chi_{[\frac{3}{4},1]}$, $\quad f_8 = \chi_{[0,\frac{1}{8}]}$, …

$f_n \to 0$ in $L^p(E)$ but $\{f_n(x)\}$ does not converge pointwise for any $x \in [0,1]$.

However, we do have the following theorem:

Theorem: Let E be a measurable set and $1 \leq p < \infty$. Suppose $\{f_n\}$ is a sequence in $L^p(E)$ that converges pointwise a.e. on E to $f \in L^p(E)$ then $f_n \to f$ in $L^p(E)$ if and only if $\lim\limits_{n\to\infty} \int_E |f_n|^p = \int_E |f|^p$.

Proof: By excising a set of measure 0 we can assume $f_n \to f$ pointwise on E.

From Minkowski's inequality (i.e. triangle inequality for $L^p(E)$)

$$\|f_n\|_p \leq \|f_n - f\|_p + \|f\|_p$$

So $$\|f_n\|_p - \|f\|_p \leq \|f_n - f\|_p.$$

So if $f_n \to f$ in $L^p(E)$ then $\lim\limits_{n\to\infty} \int_E |f_n|^p = \int_E |f|^p$.

Now let's assume $\lim_{n\to\infty} \int_E \ |f_n|^p = \int_E \ |f|^p$.

Define $\varphi(t) = |t|^p$ for all t.

Since $\varphi''(t) \geq 0$ for $t \neq 0$, φ is convex. Thus

$\varphi(\frac{a+b}{2}) \leq \frac{\varphi(a)+\varphi(b)}{2}$ for all a, b.

Thus we have:

$0 \leq \frac{|a|^p+|b|^p}{2} - \left|\frac{a-b}{2}\right|^p$ for all a, b.

Define $h_n(x) = \frac{|f_n(x)|^p+|f(x)|^p}{2} - \left|\frac{f_n(x)-f(x)}{2}\right|^p$ for all $x \in E$.

Since $f_n \to f$ pointwise on E; $h_n \to |f|^p$ pointwise on E and $h_n(x) \geq 0$.

By Fatou's lemma: $\int_E \ |f|^p \leq liminf \int_E \ h_n$

$$= liminf \int_E \ \left(\frac{|f_n(x)|^p+|f(x)|^p}{2} - \left|\frac{f_n(x)-f(x)}{2}\right|^p\right)$$

$$= \int_E \ |f|^p - limsup \int_E \ \left|\frac{f_n(x)-f(x)}{2}\right|^p$$

since $\lim_{n\to\infty} \int_E \ |f_n|^p = \int_E \ |f|^p$.

Thus $limsup \int_E \ \left|\frac{f_n(x)-f(x)}{2}\right|^p \leq 0.$

So $\lim_{n\to\infty} \int_E \ |f_n - f|^p = 0$ and $f_n \to f$ in $L^p(E)$.

Notice that in our example of $f_n \to f$ pointwise, but not in $L^p(0,1)$,

$\lim_{n\to\infty} \int_0^1 |f_n|^p \neq \int_0^1 |f|^p$.

The Borel-Cantelli Lemma says that if $\{E_k\}_1^\infty$ is a countable collection of measurable sets with $\sum_{k=1}^\infty m(E_k) < \infty$ then almost all $x \in \mathbb{R}$ belong to at most finitely many $E's$.

Thus there is a set E_0 with $m(E_0) = 0$ and if $x \in E{\sim}E_0$ then there is some $K(x)$ such that if $k \geq K(x)$ then

$$|f_{k+1}(x) - f_k(x)| \leq \epsilon_k.$$

Let $x \in E{\sim}E_0$. Then we have:

$$|f_{n+k}(x) - f_n(x)| \leq \sum_{j=n}^{n+k-1} |f_{j+1}(x) - f_j(x)|$$

$$\leq \sum_{j=n}^{\infty} \epsilon_j \;; \quad \text{for all } n \geq K(x) \text{ and all } k.$$

Since $\sum_{j=1}^\infty \epsilon_j$ converges, the sequence of real numbers $\{f_k(x)\}$ is Cauchy.

Since the real numbers are complete: $f_k(x) \to f(x)$, a real number.

Define $f(x) = 0$ on E_0. Thus f is defined on E.

Since $\|f_{k+1} - f_k\|_p \leq \epsilon_k^2$ for all k

$\|f_{n+k} - f_k\|_p \leq \sum_{j=n}^\infty \epsilon_j^2$ or equivalently:

$\int_E |f_{n+k} - f_n|^p \leq (\sum_{j=n}^\infty \epsilon_j^2)^p$ for all n, k.

Since $f_n \to f$ pointwise a.e. on E, take the limit as $k \to \infty$. By Fatou's lemma we get:

$$\int_E |f - f_n|^p \leq liminf \int_E |f_{n+k} - f_n|^p \leq (\sum_{j=n}^{\infty} \epsilon_j^2)^p \quad \text{for all } n.$$

Since $\sum_{k=1}^{\infty} {\epsilon_k}^2$ converges we have $\lim_{n\to\infty} \sum_{j=n}^{\infty} {\epsilon_n}^2 = 0.$

Thus we have:

$$\lim_{n\to\infty} \int_E |f - f_n|^p = 0 \text{ and } f_n \to f \text{ in } L^p(E).$$

Theorem: Let E be a measurable set and $1 \leq p \leq \infty$. Then every rapidly Cauchy sequence in $L^p(E)$ converges both with respect to the $L^p(E)$ norm and pointwise a.e. on E to a function in $L^p(E)$.

Proof: Assume $1 \leq p < \infty$ ($p = \infty$ is left as an exercise).

Let $\{f_n\}$ be a rapidly Cauchy sequence in $L^p(E)$.

By possibly excising a set of measure 0 we can assume that the $f_n{}'s$ are real valued.

Choose $\sum_{k=1}^{\infty} \epsilon_k$ such that:

$$\|f_{k+1} - f_k\|_p \leq \epsilon_k^2 \text{ for all } k.$$

Thus $\int_E |f_{k+1} - f_k|^p \leq \epsilon_k^{2p}$ for all k.

Fix $k \in \mathbb{Z}^+$.

$|f_{k+1}(x) - f_k(x)| \geq \epsilon_k$ if and only if $|f_{k+1}(x) - f_k(x)|^p \geq \epsilon_k^p$.

By Chebychev's inequality:

$$\begin{aligned} m\{x \in E|\ |f_{k+1}(x) - f_k(x)| \geq \epsilon_k\} &= m\{x \in E|\ |f_{k+1}(x) - f_k(x)|^p \geq \epsilon_k^p\} \\ &\leq \frac{1}{\epsilon_k^p} \int_E \left|f_{k+1}(x) - f_k(x)\right|^p \\ &\leq \epsilon_k^p . \end{aligned}$$

Since $p \geq 1$, $\sum_1^\infty \epsilon_k^p$ converges.

Let $E_k = \{x \in E|\ |f_{k+1}(x) - f_k(x)| \geq \epsilon_k\}$.

Since $\sum_1^\infty \epsilon_k^p$ converges, $\sum_{k=1}^\infty m(E_k)$ converges.