# $L^p$ Spaces

Def: For $E$ a measurable set, $1 < p < \infty$, and a function $f \in L^p(E)$, define:

$$\|\boldsymbol{f}\|_{\boldsymbol{p}} = (\int_E |f|^p)^{\frac{1}{p}}.$$

The functional $\|\cdot\|_p$ is a norm on $L^p(E)$.

It's clear that $\|\lambda f\|_p = |\lambda|\|f\|_p$, and $\|f\|_p \geq 0$ with $\|f\|_p = 0$ if, and only if, $f = 0$ a.e. on $E$.

What is less obvious is the triangle inequality:

$$\|f + g\|_p \leq \|f\|_p + \|g\|_p.$$

This is called the **Minkowski inequality**.

Def. The **conjugate** of a number $p \in (1, \infty)$ is the number $q = \frac{p}{p-1}$, which is the unique $q \in (1, \infty)$ for which:

$$\frac{1}{p} + \frac{1}{q} = 1.$$

The conjugate of $1$ is defined to be $\infty$, and the conjugate of $\infty$ is defined to be $1$.

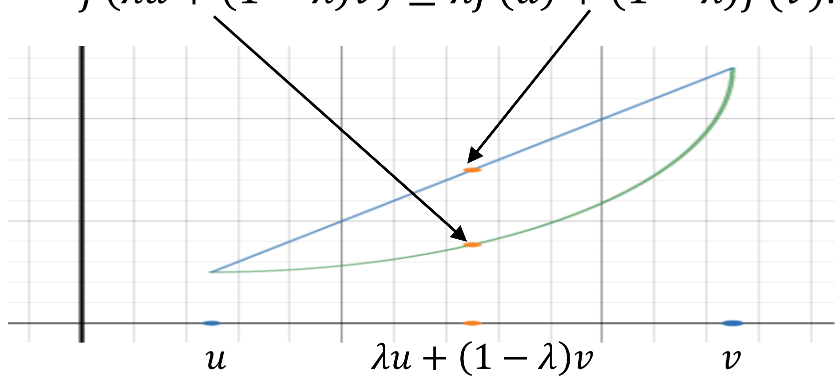
**Young's inequality:** for $1 < p < \infty$, $q$ the conjugate of $p$, and any two positive numbers $a, b$,

$$ab \leq \frac{a^p}{p} + \frac{b^p}{q}.$$

Proof: $f(x) = e^x$ has a positive second derivative and therefore is convex, i.e. for any $\lambda \in [0,1]$, and any numbers $u, v$.

$$e^{\lambda u+(1-\lambda)v} \leq \lambda e^u + (1-\lambda)e^v$$

i.e. $$f(\lambda u + (1-\lambda)v) \leq \lambda f(u) + (1-\lambda)f(v).$$




In particular, setting $\lambda = \frac{1}{p}$ , $1-\lambda = \frac{1}{q}$, $u = ln a^p$, $v = ln b^q$

$$e^{(\frac{1}{p}\ln a^p + \frac{1}{q}\ln b^q)} \leq \frac{1}{p}e^{(\ln a^p)} + \frac{1}{q}e^{(\ln b^q)}$$

$$ab \leq \frac{1}{p}a^p + \frac{1}{q}b^q.$$

Def. $\boldsymbol{sgn(f)} = 1$ if $f(x) \geq 0$ and $-1$ if $f(x) < 0$.

Theorem: Let $E$ be a measurable set, $1 \leq p < \infty$, and $q$ the conjugate of $p$. If $f \in L^p(E)$ and $g \in L^q(E)$, then $f \cdot g$ is integrable over $E$ and,

$$\int_E |fg| \leq \|f\|_p \|g\|_q \quad \textbf{(Holder's inequality)}$$

Moreover, if $f \neq 0$, the function:

$$f^* = \|f\|_p^{1-p} \cdot sgn(f) \cdot |f|^{p-1} \in L^q(E)$$

and $\int_E f \cdot f^* = \|f\|_p$ and $\|f^*\|_q = 1$.

Proof: First let $p = 1$ then $g \in L^\infty(E)$.

So, $\|g\|_\infty =$ essential upper bound of $g$ on $E$.

$$\int_E |fg| \leq \|g\|_\infty \int_E |f| = \|g\|_\infty \|f\|_1.$$

If $p = 1$, $f^* = sgn(f)$ so $f \cdot f^* = |f|$ and

$$\int_E f \cdot f^* = \int_E |f| = \|f\|_1 \text{ and } \|f^*\|_\infty = 1.$$

If $p > 1$, assume $f \not\equiv 0, g \not\equiv 0$ else there is nothing to prove.

Notice that if Holder's inequality is true for $f$ replaced by $\frac{f}{\|f\|_p}$ and $g$ replaced by $\frac{g}{\|g\|_q}$ then it's true for $f$ and $g$ as well since:

$$\int_E \left|\frac{f}{\|f\|_p}\frac{g}{\|g\|_q}\right| \leq \left\|\frac{f}{\|f\|_p}\right\|_p \left\|\frac{g}{\|g\|_q}\right\|_q = 1,$$

If and only if $\int_E |fg| \leq \|f\|_p\|g\|_q$.

Thus we can assume $\|f\|_p = \|g\|_q = 1$, that is:

$$\int_E |f|^p = 1 \text{ and } \int_E |g|^q = 1.$$

In which case Holder's inequality becomes: $\int_E |fg| \leq 1$.

Since $|f|^p$ and $|g|^q$ are integrable over $E$, $f$ and $g$ are finite a.e. on $E$. By Young's inequality we have:

$$|f \cdot g| = |f||g| \leq \frac{|f|^p}{p} + \frac{|g|^q}{q} \text{ a.e. on } E.$$

By the integral comparison test $fg$ is integrable over $E$ and,

$$\int_E |fg| \leq \frac{1}{p}\int_E |f|^p + \frac{1}{q}\int_E |g|^q = \frac{1}{p} + \frac{1}{q} = 1.$$

Thus Holder's inequality is proved.

Now notice that since $f^* = \|f\|_p^{1-p} sgn(f)|f|^{p-1}$,

$$f \cdot f^* = \|f\|_p^{1-p}|f|^p \qquad \text{a.e. on } E.$$

So $\int_E |f \cdot f^*| = \|f\|_p^{1-p} \int_E |f|^p = \|f\|_p^{1-p}\|f\|^p = \|f\|_p.$

And since $q(p-1) = \frac{p}{p-1}(p-1) = p$,

$$\|f^*\|_q = (\int_E |f^*|^q)^{\frac{1}{q}} = (\int_E \|f\|_p^{(1-p)q}|f|^{(p-1)q})^{\frac{1}{q}}$$

$$= (\|f\|_p^{(1-p)q})^{\frac{1}{q}}(\int_E |f|^{(p-1)q})^{\frac{1}{q}}$$

$$= (\|f\|_p^{-p})^{\frac{1}{q}}(\int_E |f|^p)^{\frac{1}{q}}$$

$$= (\int_E |f|^p)^{-\frac{1}{q}}(\int_E |f|^p)^{\frac{1}{q}} = 1.$$

If $f \in L^p(E),\ f \not\equiv 0$, we call $f^* = \|f\|_p^{1-p} sgn(f)|f|^{p-1}$ **the conjugate function of $f$.**

**The Minkowski inequality**: Let $E$ be a measurable set and $1 \leq p \leq \infty$.

If $f, g \in L^p(E)$, then $f + g \in L^p(E)$ and:

$$\|f + g\|_p \leq \|f\|_p + \|g\|_p.$$

Proof: We have already seen this is true for $p = 1$ and $p = \infty$.

Assume $1 < p < \infty$.

Since $|f(x) + g(x)|^p \leq 2^p[|f(x)|^p + |g(x)|^p]$

we know $f + g \in L^p(E)$.

If $f + g \not\equiv 0$, then by Holder's inequality:

$$\|f + g\|_p = \int_E (f + g)(f + g)^*$$

$$= \int_E f(f + g)^* + \int_E (g(f + g)^*$$

$$\leq \|f\|_p \|(f + g)^*\|_q + \|g\|_p \|(f + g)^*\|_q$$

But $\|(f + g)^*\|_q = 1$ so

$$= \|f\|_p + \|g\|_q.$$

The special case of Holder's inequality where $p = q = 2$ is called the **Cauchy-Schwarz inequality**.

Cauchy-Schwarz inequality: Let $E$ be measurable and $f, g \in L^2(E)$, then $f \cdot g \in L^1(E)$ and

$$\int_E |f \cdot g| \leq \left(\sqrt{\int_E f^2}\right)\left(\sqrt{\int_E g^2}\right).$$

This is also the analogue of the Cauchy-Schwarz inequality for vectors in $\mathbb{R}^n$, i.e. $\vec{v}, \vec{w} \in \mathbb{R}^n$: $\quad |\vec{v} \cdot \vec{w}| \leq \|\vec{v}\|\|\vec{w}\|$.

Ex. Prove the if $f \in L^2[a,b]$, then $\int_a^b |f| \leq (\sqrt{b-a})(\int_a^b |f|^2)^{\frac{1}{2}}$ and thus $L^2[a,b] \subseteq L^1[a,b]$.

By the Cauchy-Schwarz inequality we have:

$$\int_a^b |1 \cdot f| \leq (\int_a^b 1)^{\frac{1}{2}} (\int_a^b |f|^2)^{\frac{1}{2}} = (\sqrt{b-a})(\int_a^b |f|^2)^{\frac{1}{2}}.$$

Although $L^2[a,b] \subseteq L^1[a,b]$, $\;L^1[a,b] \subsetneq L^2[a,b]$.

For example, $f(x) = \frac{1}{\sqrt{x}} \in L^1[0,1]$, but $f(x) = \frac{1}{\sqrt{x}} \notin L^2[0,1]$.

Corollary: Let $E$ be a measurable set and $1 < p < \infty$. Suppose $F$ is a family of functions in $L^p(E)$ that is bounded in $L^p(E)$, i.e. there is a constant $M \geq 0$ such that:

$$\|f\|_p \leq M \text{ for all } f \in F.$$

Then the family $F$ is uniformly integrable over $E$.

Proof: Let $\epsilon > 0$.

We must show there exists a $\delta > 0$ such that for any $f \in F$, if $A \subseteq E$ is measurable and $m(A) < \delta$ then $\int_A |f| < \epsilon$.

Let $A \subseteq E$ be measurable and $m(A) < \infty$.

Let $g(x) = 1$ for $x \in A$. Then $g \in L^q(A)$.

Since $f \in L^p(E)$, it's restriction to $A$ is in $L^p(A)$.

By Holder's inequality:

$$\int_A |f| = \int_A |f|\, g \leq (\int_A |f|^p)^{\frac{1}{p}} \cdot (\int_A |g|^q)^{\frac{1}{q}}.$$

But for all $f \in F$:

$$(\int_A |f|^p)^{\frac{1}{p}} \leq \left(\int_E |f|^p\right)^{\frac{1}{p}} \leq M \text{ and } (\int_A |g|^q)^{\frac{1}{q}} = (m(A))^{\frac{1}{q}}.$$

So, $\int_A |f| \leq M(m(A))^{\frac{1}{q}}$.

Let $\delta = (\frac{\epsilon}{M})^q$.

Then $m(A) < (\frac{\epsilon}{M})^q$ and $\int_A |f| < M \cdot (\left(\frac{\epsilon}{M}\right)^q)^{\frac{1}{q}} = \epsilon$.

Corollary: Let $E$ be a measurable set of finite measure and

$1 \leq p_1 < p_2 \leq \infty.$

Then, $L^{p_2}(E) \subseteq L^{p_1}(E)$

and $\|f\|_{p_1} \leq c\|f\|_{p_2}$ for all $f \in L^{p_2}(E)$

where $c = [m(E)]^{(\frac{p_2-p_1}{p_2 p_1})}$ if $p_2 < \infty$ and

$c = [m(E)]^{(\frac{1}{p_1})}$ if $p_2 = \infty.$

Proof: Assume $p_2 < \infty.$

Define $p = \frac{p_2}{p_1} > 1$ and let $q$ be the conjugate of $p$.

Let $f \in L^{p_2}(E)$. So $\int_E |f|^{p_2} < \infty.$

Notice that $f^{p_1} \in L^p(E)$ since $\int_E |f^{p_1}|^p = \int_E |f|^{p_2} < \infty.$

And $g = \chi_E$ belongs to $L^q(E)$ because $m(E) < \infty.$

By the Holder inequality:

$$\int_E |f|^{p_1} = \int_E |f|^{p_1} \cdot g \leq \|f^{p_1}\|_p \|g\|_q$$

$$= \left(\int_E |f^{p_1}|^{\frac{p_2}{p_1}}\right)^{\frac{p_1}{p_2}} \left(\int_E |g|^q\right)^{\frac{1}{q}}$$

$$= [\left(\int_E |f|^{p_2}\right)^{\frac{1}{p_2}}]^{p_1} (m(E))^{\frac{1}{q}}$$

$$= \|f\|_{p_2}^{p_1} (m(E))^{\frac{1}{q}}.$$

So: $$\|f\|_{p_1} = \left(\int_E |f|^{p_1}\right)^{\frac{1}{p_1}} \leq \|f\|_{p_2}(m(E))^{\frac{1}{q}(\frac{1}{p_1})}$$

$$p = \frac{p_2}{p_1}, \qquad \text{and } \frac{1}{p} + \frac{1}{q} = 1$$

$$\text{so } \frac{1}{p} = 1 - \frac{p_1}{p_2} \qquad \text{and } \frac{1}{q} \cdot \frac{1}{p_1} = \frac{p_2 - p_1}{p_1 p_2}$$

$$\|f\|_{p_1} \leq \|f\|_{p_2}(m(E))^{(\frac{p_2-p_1}{p_1 p_2})}.$$

If $p_2 = \infty$ and $f \in L^\infty(E)$ then

$$\|f\|_{p_1} = \left(\int_E |f|^{p_1}\right)^{\frac{1}{p_1}} \leq \left[(\|f\|_\infty)^{p_1} \int_E 1\right]^{\frac{1}{p_1}}$$

$$= \|f\|_\infty (m(E))^{\frac{1}{p_1}}.$$

Ex. Show that If $E = [0,1]$ and $1 \leq p_1 < p_2 \leq \infty$, $L^{p_2}(E)$ is a proper subspace of $L^{p_1}(E)$.

$m(E) < \infty$ so $L^{p_2}(E) \subseteq L^{p_1}(E)$.

Let $f(x) = x^\alpha$, $0 < x \leq 1$, where $-\frac{1}{p_1} < \alpha \leq -\frac{1}{p_2}$,

then $f(x) \in L^{p_1}(E) \sim L^{p_2}(E)$.

For example, if $p_1 = 1$, $p_2 = 2$; $-1 < \alpha \leq -\frac{1}{2}$

then $f(x) = x^\alpha$ is $L^1((0,1])$ since $\int_0^1 x^\alpha = \frac{1}{1+\alpha}$, but $\int_0^1 x^{2\alpha} = \infty$.

Ex. In general, if $m(E) = \infty$ there are no inclusion relations among $L^p(E)$ spaces. For example, if $E = (0, \infty)$ and $f(x) = \frac{x^{-\frac{1}{2}}}{1+|\ln x|}$ $f \in L^p(E)$ if, and only if, $p = 2$.