L^p Spaces- HW Problems

1. Find functions f and g such that $f \in L^2(0,\infty)$, but $f \notin L^1(0,\infty)$, and $g \in L^1(0,\infty)$, but $g \notin L^2(0,\infty)$.

2. Find functions f and g such that $f \in L^\infty(0,\infty)$, but $f \notin L^1(0,\infty)$, and $g \in L^1(0,\infty)$, but $g \notin L^\infty(0,\infty)$. (Do not use any examples that are in the class notes for this section).

3. Prove $\left| \int_0^{2\pi} \frac{\cos(x)}{\sqrt{x^2+1}} dx \right| \leq \sqrt{\pi \tan^{-1}(2\pi)}$.

4. Either prove the following statements or find a counterexample to show that it's false.
   1. If $f, g \in L^1(0,1)$ then $fg \in L^1(0,1)$.
   2. If $f, g \in L^2(0,1)$ then $fg \in L^1(0,1)$.
   3. If $f, g \in L^\infty(0,1)$ then $fg \in L^1(0,1)$.

5. Suppose that $\{g_n\}$ is bounded in $L^1(0,1)$. Is $\{g_n\}$ uniformly integrable over $(0,1)$?

6. Show that if f is bounded on E and $f \in L^p(E)$ then $f \in L^{p'}(E)$ for any $p' > p$.

7. Show that $f(x) = \ln(\frac{1}{x})$ satisfies $f \in L^p(0,1)$ for $1 \leq p < \infty$ but $f \notin L^\infty(0,1)$.