Normed Linear Spaces

Let E be a measurable set of real numbers.

Let F be the collection of measurable extended real valued function on E that are finite a.e. on E.

Let's define two functions in F to be equivalent, $f \cong g$, if $f(x) = g(x)$ for almost all $x \in E$.

$\cong$ is an equivalence relation (i.e. it's reflexive, symmetric, and transitive).

So we can partition F into $F/_{\cong}$, equivalence classes of functions.

Notice that $F/_{\cong}$ is a linear space: $\alpha f + \beta g \in F$ if $f, g \in F$ and $\alpha, \beta \in \mathbb{R}$, and the zero element is the class of functions that are 0 a.e. on E.

Def. $\boldsymbol{L^p(E)} = \{f \in F/_{\cong} \mid \int_E |f|^p < \infty\}; \quad 1 \le p < \infty.$

Notice that $L^p(E)$ is a linear subspace of $F/_{\cong}$ since for any $a, b \in \mathbb{R}$:

$$|a + b| \le |a| + |b| \le 2\max\{|a|, |b|\}$$

So $$|a + b|^p \le 2^p(\max\{|a|, |b|\})^p \le 2^p(|a|^p + |b|^p).$$

Thus if $f, g \in L^p(E)$ then $\alpha f \in L^p(E)$ because

$$\int_E |\alpha f|^p = |\alpha|^p \int_E |f|^p < \infty.$$

And $f + g \in L^p(E)$ because $|f + g|^p \le 2^p(|f|^p + |g|^p)$

So $\int_E |f + g|^p \le \int_E 2^p(|f|^p + |g|^p) = 2^p(\int_E |f|^p + \int_E |g|^p) < \infty.$

So $\alpha f + \beta g \in L^p(E)$.

Clearly $f = 0$ a.e. on E is also in $L^p(E)$.

Notice that $L^1(E)$ is just the integrable functions over E.

Def. $f \in F$ is called **essentially bounded** if there is come $M \geq 0$ such that $|f(x)| \leq M$ for almost all $x \in E$. M is called an essential upper bound for f.

Ex. $f(x) = \frac{1}{x}$ if $x \in \mathbb{Q}$, $x \neq 0$

$= 2$ if $x \notin \mathbb{Q}$, $x \neq 0$

Is essentially bounded on $(-\infty, \infty)$ because $|f(x)| \leq 2$ a.e..

Ex. $f(x) = \frac{1}{x}$ if $x \notin \mathbb{Q}$, $x \neq 0$

$= 2$ if $x \in \mathbb{Q}$, $x \neq 0$

Is not essentially bounded on $(-\infty, \infty)$.

Def. $\boldsymbol{L^\infty(E)}$ is the set of essentially bounded function on E.

$L^\infty(E)$ is also a linear subspace of ${}^{F}/_{\cong}$.

Def. Real valued functions whose domain is a linear space such as $L^p(E)$ are called **functionals.**

Ex. We can define a functional T on $L^1[0,1]$ by

$$T: L^1[0,1] \to \mathbb{R} \text{ by } T(f) = \int_0^1 |f|.$$

Def. Let X be a linear space. A real valued function $\|\cdot\|$ on X is called a **norm** if for each $f, g \in X$ and $\alpha \in \mathbb{R}$:

1. $\|f\| \geq 0$ and $\|f\| = 0$ if and only if $f = 0$.
2. $\|\alpha f\| = |\alpha|\|f\|$ (positive homogeneity)
3. $\|f + g\| \leq \|f\| + \|g\|$ (Triangle inequality).

If X is a normed linear space, $f \in X$ is called a **unit function** if $\|f\| = 1$. Any $f \not\equiv 0$, can be normalized, i.e. turned into a normal function, by taking $\frac{f}{\|f\|}$.

Ex. $\mathbb{R}^n$ is a normed linear space with $v =< a_1, \ldots, a_n >$; and $\|v\| = \sqrt{a_1^2 + a_2^2 + \cdots + a_n^2}$.

Ex. $L^1(E)$ is a normed linear space with $\|f\|_1 = \int_E |f|$.

Notice that:

$\|f\|_1 = \int_E |f| \geq 0$ and $\|f\|_1 = \int_E |f| = 0$ if and only if $f = 0$ a.e. on E.

$\|\lambda f\|_1 = \int_E |\lambda f| = |\lambda| \int_E |f| = |\lambda| \, \|f\|_1$

$|f + g| \leq |f| + |g|$ a.e on E so

$\|f + g\|_1 = \int_E |f + g| \leq \int_E |f| + \int_E |g| = \|f\|_1 + \|g\|_1$

Ex. $L^\infty(E)$ is a normed linear space with

$\|f\|_\infty = \inf\{essential\ upper\ bounds\ of\ f\}$. This is called the **essential supremum** of f.

Nonnegativity and homogeneity follow as in the previous example .

To prove the triangle inequality we first show $|f| \leq \|f\|_\infty$ a.e. on E.

For each n, there is a subset $E_n \subseteq E$ with $|f| \leq \|f\|_\infty + \frac{1}{n}$ on $E{\sim}E_n$ and $m(E_n) = 0$.

Let $E_\infty = \bigcup_{n=1}^{\infty} E_n$, then $|f| \leq \|f\|_\infty$ on $E{\sim}E_\infty$ and $m(E_\infty) = 0$.

Thus $|f| \leq \|f\|_\infty$ a.e. on E.

So we have:

$$|f+g| \leq |f| + |g| \leq \|f\|_\infty + \|g\|_\infty \text{ a.e. on } E.$$

So $\|f\|_\infty + \|g\|_\infty$ is an essential upper bound for $f + g$ thus

$$\|f+g\|_\infty \leq \|f\|_\infty + \|g\|_\infty.$$

Ex. For $1 \leq p < \infty$ define $\boldsymbol{l^p}$ to be the collection of real sequences

$a = (a_1, a_2, a_3, \dots)$ for which $\sum_{k=1}^{\infty} |a_k|^p < \infty$. l^p is a linear space.

Suppose that if $a = (a_1, a_2, a_3, \dots)$ and $b = (b_1, b_2, b_3, \dots)$ such that $\sum_{k=1}^{\infty} |a_k|^p < \infty$ and $\sum_{k=1}^{\infty} |b_k|^p < \infty$.

then $a + b = (a_1 + b_1, a_2 + b_2, \dots)$.

To see that $a + b \in l^p$ notice that:

$$|a_i + b_i|^p \leq 2^p(|a_i|^p + |b_i|^p).$$

Which means

$\sum_{k=1}^{\infty} |a_k + b_k|^p \leq 2^p \sum_{k=1}^{\infty}(|a_k|^p + |b_k|^p) < \infty$.

$\lambda a = (\lambda a_1, \lambda a_2, \lambda a_3, \dots)$;

$\sum_{k=1}^{\infty} |\lambda a_k|^p = |\lambda|^p \sum_{k=1}^{\infty} |a_k|^p < \infty$

so $\lambda a \in l^p$.

The zero sequence, $a = (0, 0, 0, \dots)$ is also in l^p, so l^p is a linear space.

We define a norm on l^1 by

$$\|\boldsymbol{a}\|_{\mathbf{1}} = \sum_{k=1}^{\infty} |a_k|.$$

We define a norm on l^p by

$$\|\boldsymbol{a}\|_{\boldsymbol{p}} = \sqrt[p]{\sum_{k=1}^{\infty} |a_k|^p}$$

We define l^∞ to be bounded sequences and the norm by:

$$\|\boldsymbol{a}\|_\infty = \sup_{1 \le k < \infty} |a_k|$$

Ex. Let $C[a,b] = \{$continuous function on $[a,b]\}$, where $[a,b]$ is a closed bounded interval. This is a normed linear space with

$$\|f\| = \max_{x \in [a,b]} |f(x)|.$$

Note: There can be more than 1 norm defined on a linear space.

For example, we could define a different norm on $C[0,1]$ by:

$$\|f\| = \int_a^b |f| \,.$$