Lebesgue Measurable Sets

The outer measure m^* has 4 important properties:

1. It's defined for all sets of real numbers.
2. $m^*(I) = l(I) = b - a$, for any interval $(a, b), [a, b), (a, b], [a, b]$.
3. $m^*(\bigcup_{k=1}^{\infty} E_k) \leq \sum_{k=1}^{\infty} m^*(E_k)$ (i.e. m^* is countably subadditive).
4. $m^*(t + E) = m^*(E)$ for any $t \in \mathbb{R}$ (i.e. m^* is translation invariant).

The problem is there are disjoint sets A, B such that:

$$m^*(A \cup B) < m^*(A) + m^*(B).$$

This does not correspond well to one's intuition about how a measure should work. To solve this problem we will simply remove these "bad" sets.

For any set $E \subseteq \mathbb{R}$, Notice that we can always write a set A as:

$$A = (A \cap E) \cup (A \cap E^c).$$

Def. A set $E \subseteq \mathbb{R}$ is said to be **measurable** provided for any set A:

$$m^*(A) = m^*(A \cap E) + m^*(A \cap E^c).$$

Notice that if E is a measurable set and D is any set with $E \cap D = \phi$,

and we take the set $A = E \cup D$, we get:

$$m^*(E \cup D) = m^*([E \cup D] \cap E) + m^*([E \cup D] \cap E^c)$$

$$= m^*(E) + m^*(D) \qquad \text{i.e. there's no inequality.}$$

If we write $A = [A \cap E] \cup [A \cap E^c]$ and A is any set (possibly not measurable) we know from the subadditivity of m^*:

$$m^*(A) \leq m^*(A \cap E) + m^*(A \cap E^c).$$

Thus E is measurable if, and only if:

$$m^*(A) \geq m^*(A \cap E) + m^*(A \cap E^c).$$

This last inequality will always hold if $m^*(A) = \infty$. So it's enough to show E is measurable by showing this inequality holds for sets A where $m^*(A)$ is finite.

The definition of a set E being measurable is symmetric in E and E^c.

Thus E is measurable if, and only if, E^c is measurable.

Prop. If $m^*(E) = 0$ then E is measurable.

Proof: Let A be any set.

$A \cap E \subseteq E$ and $A \cap E^c \subseteq A$ thus,

$0 \leq m^*(A \cap E) \leq m^*(E) = 0$ and $m^*(A \cap E^c) \leq m^*(A)$.

Thus we have:

$$m^*(A) \geq m^*(A \cap E^c) = 0 + m^*(A \cap E^c) = m^*(A \cap E) + m^*(A \cap E^c)$$

Hence E is measurable.

Prop. Let $E_1, \ldots, E_n$ be measurable sets then $\bigcup_{i=1}^{n} E_k$ is a measurable set.

Proof: Let's start by showing this for two measurable sets E_1 and E_2.

Let A be any set.

Since E_1 is measurable:

$$m^*(A) = m^*(A \cap E_1) + m^*(A \cap {E_1}^c).$$

Since E_2 is measurable:

$$m^*(A \cap {E_1}^c) = m^*([A \cap {E_1}^c] \cap E_2) + m^*([A \cap {E_1}^c] \cap {E_2}^c)$$

Thus:

$$m^*(A) = m^*(A \cap E_1) + m^*([A \cap {E_1}^c] \cap E_2) + m^*([A \cap {E_1}^c] \cap {E_2}^c)$$

Now using the identities:

$$[A \cap {E_1}^c] \cap {E_2}^c = A \cap [E_1 \cup E_2]^c$$

$$(A \cap E_1) \cup [A \cap {E_1}^c \cap E_2] = A \cap [E_1 \cup E_2]$$

we get:

$$\begin{aligned} m^*(A) &= m^*(A \cap E_1) + m^*([A \cap {E_1}^c] \cap E_2) + m^*(A \cap [E_1 \cup E_2]^c) \\ &\geq m^*([A \cap E_1] \cup [A \cap {E_1}^c \cap E_2]) + m^*(A \cap [E_1 \cup E_2]^c) \\ &= m^*(A \cap [E_1 \cup E_2]) + m^*(A \cap [E_1 \cup E_2]^c). \end{aligned}$$

Thus $E_1 \cup E_2$ is measurable.

To prove for a finite union use mathematical induction.

For $n = 1$, by assumption the set is measurable.

If we assume the statement is true for $n - 1$:

$$\bigcup_{i=1}^{n} E_k = [\bigcup_{i=1}^{n-1} E_k] \cup E_n.$$

Just let one set be $\bigcup_{i=1}^{n-1} E_k$ and the second set be E_n and the previous proof shows $\bigcup_{i=1}^{n} E_k$ is measurable.

Prop. Let A be any set and $\{E_k\}_{k=1}^{n}$ a finite disjoint collection of measurable sets. Then:

$$m^*(A \cap [\bigcup_{k=1}^{n} E_k]) = \sum_{k=1}^{n} m^*(A \cap E_k).$$

Thus if $A = \bigcup_{k=1}^{n} E_k$ then,

$$m^*(\bigcup_{k=1}^{n} E_k) = \sum_{k=1}^{n} m^*(E_k).$$

Proof: By induction, if $n = 1$ then the conclusion is

$m^*(A \cap E) = m^*(A \cap E)$, which is clearly true.

Assume the statement is true for $n - 1$ and prove it's true for n:

We know that E_n is measurable so we can say,

$$m^*(A \cap \bigcup_{k=1}^{n} E_k) = m^*([A \cap \bigcup_{k=1}^{n} E_k] \cap E_n) + m^*([A \cap \bigcup_{k=1}^{n} E_k] \cap {E_n}^c)$$

(Think of the arbitrary set A in the definition of E_n being measurable as being the set $A \cap \bigcup_{k=1}^{n} E_k$).

Notice that since $\{E_k\}_{k=1}^{n}$ are disjoint:

$$A \cap [\bigcup_{k=1}^{n} E_k] \cap E_n = A \cap E_n$$

and
$$A \cap [\bigcup_{k=1}^{n} E_k] \cap {E_n}^c = A \cap [\bigcup_{k=1}^{n-1} E_k]$$

so
$$m^*(A \cap \bigcup_{k=1}^{n} E_k) = m^*(A \cap E_n) + m^*(A \cap \bigcup_{k=1}^{n-1} E_k).$$

By induction we know, $m^*(A \cap \bigcup_{k=1}^{n-1} E_k) = \sum_{k=1}^{n-1} m^*(A \cap E_k)$

so
$$m^*(A \cap \bigcup_{k=1}^{n} E_k) = m^*(A \cap E_n) + \sum_{k=1}^{n-1} m^*(A \cap E_k)$$
$$= \sum_{k=1}^{n} m^*(A \cap E_k).$$

Def. A collection of subsets of $\mathbb{R}$ is called an **algebra** if it contains $\mathbb{R}$ and is closed under the formation of complements and finite unions.

Thus, the set of measurable sets in an algebra.

Notice also that any collection of sets that are closed under complements and finite unions is closed under finite intersections.

The union of a countable collection of measurable sets can be written as a union of a countable collection of disjoint measurable sets. Let $\{A_k\}_{k=1}^{\infty}$ be a countable collection of measurable sets:

Let $A_1' = A_1$ and $A_k' = A_k \sim \bigcup_{i=1}^{k-1} A_i$.

Since the collection of measurable sets is an algebra, $\{{A_k}'\}_{k=1}^{\infty}$ are also measurable sets (but are disjoint) and $\bigcup_{k=1}^{\infty} A_k = \bigcup_{k=1}^{\infty} {A_k}'$.

Prop. If $\{E_k\}_{k=1}^{\infty}$ is a countable collection of measurable sets then $\bigcup_{i=1}^{\infty} E_i$ is measurable.

Proof: Let E be a countable union of measurable sets. We can write:

$E = \bigcup_{i=1}^{\infty} E_i$ where the $\{E_i\}_{i=1}^{\infty}$ are disjoint.

Let A be any set.

Let $B_n = \bigcup_{k=1}^{n} E_k$.

B_n is measurable because it's a finite union of measurable sets and ${B_n}^c \supseteq E^c$ so:

$$m^*(A) = m^*(A \cap B_n) + m^*(A \cap {B_n}^c) \geq m^*(A \cap B_n) + m^*(A \cap E^c).$$

Since $\{E_i\}_{i=1}^{\infty}$ are disjoint:

$$m^*(A \cap B_n) = \sum_{k=1}^{n} m^*(A \cap E_k)$$

thus, $m^*(A) \geq \sum_{k=1}^{n} m^*(A \cap E_k) + m^*(A \cap E^c)$, for all n.

Hence: $$m^*(A) \geq \sum_{k=1}^{\infty} m^*(A \cap E_k) + m^*(A \cap E^c).$$

By countable subadditivity we have,

$$m^*(A) \geq m^*(A \cap E) + m^*(A \cap E^c).$$

Thus, E is measurable.

Def. A collection of subsets of $\mathbb{R}$ is called a **$\boldsymbol{\sigma}$-algebra** if it contains $\mathbb{R}$ and is closed under the formation of complements and countable unions.

Thus the collection of measurable sets is a σ-algebra.

By the laws of set theory, if a collection of sets is closed under complements and countable unions then it's closed under countable intersections.

Prop. Every interval is measurable.

Proof: To show every interval is measurable we only need to show intervals of the form (a, ∞) are measurable because measurable sets are a σ-algebra (Thus, every interval can be created from (a, ∞) via complements and countable unions.).

Let A be any set. Assume $a \notin A$, otherwise replace A by $A \sim \{a\}$.

We must show:

$$m^*(A_1) + m^*(A_2) \leq m^*(A)$$

where $A_1 = A \cap (-\infty, a)$ and $A_2 = A \cap (a, \infty)$.

Since $m^*(A)$ is an infimum, we just need to show that for any countable collection $\{I_k\}_{k=1}^{\infty}$ of open bounded intervals that covers A (we can assume $m^*(A)$ is finite, so each I_k is bounded):

$$m^*(A_1) + m^*(A_2) \leq \sum_{k=1}^{\infty} l(I_k).$$

Define $I_{k,1} = I_k \cap (-\infty, a), \quad I_{k,2} = I_k \cap (a, \infty)$.

Then, $l(I_k) = l\left(I_{k,1}\right) + l(I_{k,2})$.

Since $\{I_{k,1}\}_{k=1}^{\infty}$ and $\{I_{k,2}\}_{k=1}^{\infty}$ are countable collections of open bounded intervals that cover A_1 and A_2 respectively

$$m^*(A_1) \leq \sum_{k=1}^{\infty} l\left(I_{k,1}\right) \text{ and } m^*(A_2) \leq \sum_{k=1}^{\infty} l\left(I_{k,2}\right).$$

Therefore: $m^*(A_1) + m^*(A_2) \leq \sum_{k=1}^{\infty} l(I_{k,1}) + \sum_{k=1}^{\infty} l(I_{k,2})$

$$= \sum_{k=1}^{\infty}(l(I_{k,1}) + l(I_{k,2}))$$

$$= \sum_{k=1}^{\infty} l\,(I_k).$$

So $m^*(A_1) + m^*(A_2) \leq m^*(A)$ and the interval is measurable.

Every open set in $\mathbb{R}$ is the disjoint union of a countable collection of open intervals. Thus, by the two previous propositions, every open set is measurable. Every closed set is the complement of an open set, thus every closed set is measurable.

Def. E is called a $\boldsymbol{G_\delta}$ **set** if it is the intersection of a countable collection of open sets. E is called an $\boldsymbol{F_\sigma}$ set if it is the union of a countable collection of closed sets.

Ex. A G_δ need not be open and an F_σ set need not be closed.

Let $A_n = (-\frac{1}{n}, \frac{1}{n}) \cup (2, 3)$.

Then $A = \bigcap_{i=1}^{\infty} A_n = \{0\} \cup (2, 3)$ is a G_δ set.

Let $B_n = \left[0, 1 - \frac{1}{n}\right]$.

Then $B = \bigcup_{n=1}^{\infty} B_n = [0, 1)$ is an F_σ set.

Since every open set or closed set is measurable and measurable sets form a σ-algebra (hence countable intersections or unions are measurable) every G_δ and F_σ set is measurable.

Ex. The set of irrational numbers, $\mathbb{R}\sim\mathbb{Q}$, is a G_δ set because:

$$\mathbb{R}\sim\mathbb{Q} = \bigcap_{i=1}^{\infty}(\mathbb{R}\sim q_i); \text{ where } \bigcup_{i=1}^{\infty} q_i = \mathbb{Q}.$$

Ex. Every closed interval is a G_δ set since:

$$[a,b] = \bigcap_{i=1}^{\infty}\left(a - \frac{1}{i},\ b + \frac{1}{i}\right).$$

Ex. Every open interval is an F_σ set since:

$$(a,b) = \bigcup_{i=1}^{\infty}\left[a + \frac{1}{i},\ b - \frac{1}{i}\right].$$

The complement of a G_δ set is an F_σ set and the complement of an F_σ set is a G_δ set.

Def. The **Borel** σ-algebra is defined to be the smallest σ-algebra of subsets of $\mathbb{R}$ containing the open sets (equivalently, it's the σ-algebra generated by open intervals in $\mathbb{R}$).

The elements of the Borel σ-algebra are called **Borel sets**.

Ex. Every G_δ and F_σ set is a Borel Set.

Since the collection of measurable sets contains all of the open sets and is a σ-algebra, it must contain all of the Borel sets as well. Thus we have:

Theorem: Every Borel set is measurable. Each interval, open set, closed set, G_δ set, and F_σ set is measurable.

Prop. The translate of a measurable set is measurable.

Proof: Let E be a measurable set. Let A be any set and $t \in \mathbb{R}$.

Since E is measurable and the outer measure is translation invariant:

$$m^*(A) = m^*(A - t) = m^*([A - t] \cap E) + m^*([A - t] \cap E^c)$$
$$= m^*(A \cap [E + t]) + m^*(A \cap [E + t]^c).$$

Therefore, $E + t$ is measurable.

Ex. Show that if a set E has positive outer measure then there is a bounded subset of E that also has positive measure.

Proof: Proof by contradiction.

Suppose every bounded subset of E has measure 0.
Let $I_k = [k, k+1]$ for $k \in \mathbb{Z}$.

Then $E = \bigcup_{k \in \mathbb{Z}} (E \cap I_k)$ and each $E \cap I_k$ is bounded and hence has $m^*(E \cap I_k) = 0$.

By the subadditivity of m^* we know:

$$0 < m^*(E) = m^*\left(\bigcup_{k \in \mathbb{Z}} (E \cap I_k)\right) \le \sum_{k \in \mathbb{Z}} m^*(E \cap I_k) = 0$$

which is a contradiction so E must have a bounded subset of positive measure.