The Fundamental Theorem of Calculus

We saw earlier that if f is a continuous function on $[a, b]$ then

$$\int_a^b Diff_h f = Av_h f(b) - Av_h f(a)$$

where $Diff_h f(x) = \frac{f(x+h)-f(x)}{h}$ and $Av_h f(x) = \frac{1}{h}\int_x^{x+h} f$.

Since f is continuous:

$\lim_{h\to 0^+} \int_a^b Diff_h f = \int_a^b f'$ if the limit exists and

$\lim_{h\to 0^+} \left(Av_h f(b) - Av_h f(a)\right) = f(b) - f(a)$.

So we get the fundamental theorem of Calculus:

$\int_a^b f' = f(b) - f(a)$.

The question is, is this statement still true even if $f(x)$ is not differentiable everywhere on (a, b)? If not, when is it true?

Theorem: Let f be absolutely continuous on the closed, bounded interval $[a, b]$. Then f is differentiable a.e. on $[a, b]$ and

$$\int_a^b f' = f(b) - f(a).$$

Proof: Since f is absolutely continuous, it is the difference of two increasing absolutely continuous function on $[a, b]$.

Therefore, by Lebesgue's theorem f is differentiable a.e. on (a, b).

Thus $\left\{Diff_{\frac{1}{n}}f\right\} = \{\frac{f\left(x+\frac{1}{n}\right)-f(x)}{\frac{1}{n}}\}$ converges pointwise a.e. on (a, b) to f'.

In addition, since f is absolutely continuous, $\left\{Diff_{\frac{1}{n}}f\right\}$ is uniformly integrable over $[a, b]$.

Since $\left\{Diff_{\frac{1}{n}}f\right\}$ is uniformly integrable over $[a, b]$ and $Diff_{\frac{1}{n}}f \to f'$ pointwise a.e. on (a, b), the Vitali Convergence Theorem says f' is integrable and

$$\lim_{n\to\infty} \int_a^b Diff_{\frac{1}{n}}f = \int_a^b \lim_{n\to\infty} Diff_{\frac{1}{n}}f = \int_a^b f'.$$

From first year Calculus we know that if f is continuous then:

$$\lim_{n\to\infty} \left(Av_{\frac{1}{n}}f(b) - Av_{\frac{1}{n}}f(a)\right) = \lim_{n\to\infty} \int_a^b Diff_{\frac{1}{n}}f$$

$$f(b) - f(a) = \int_a^b f'.$$

Def. ***f* is the indefinite integral of *g*** over $[a, b]$ if g is Lebesgue integrable over $[a, b]$ and $f(x) = f(a) + \int_a^x g$ for $x \in [a, b]$.

Theorem: A function f on a closed, bounded interval $[a, b]$ is absolutely continuous on $[a, b]$ if and only if it is an indefinite integral over $[a, b]$.

Proof: Suppose f is absolutely continuous on $[a, b]$.

Then f is absolutely continuous over $[a, x]$, for $x \in [a, b]$.

By the previous theorem : $f(x) = f(a) + \int_a^x f'$.

Thus f is an indefinite integral over $[a, b]$.

Now assume f is an indefinite integral over $[a, b]$.

Given disjoint open intervals $\{(a_k, b_k)\}_{k=1}^n$, let $E = \cup_{k=1}^n (a_k, b_k)$.

Thus we have:

$$\sum_{k=1}^n |f(b_k) - f(a_k)| = \sum_{k=1}^n \left| \int_{a_k}^{b_k} g \right| \le \sum_{k=1}^n \int_{a_k}^{b_k} |g| = \int_E |g|.$$

Let $\epsilon > 0$.

Since g is integrable over $[a, b]$, we know there is a $\delta > 0$ such that $\int_E |g| < \epsilon$ If $E \subseteq [a, b]$ and $m(E) < \delta$.

Thus f is absolutely continuous on $[a, b]$.

Lemma: Let f be integrable over the closed, bounded interval $[a,b]$. Then $f(x)=0$ a.e. on $[a,b]$ if and only if $\int_{x_1}^{x_2} f = 0$ for all $[x_1,x_2] \subseteq (a,b)$.

Proof: If $f(x)=0$ a.e. on $[a,b]$ then clearly $\int_{x_1}^{x_2} f = 0$ for all $[x_1,x_2] \subseteq (a,b)$.

Now suppose $\int_{x_1}^{x_2} f = 0$ for all $[x_1,x_2] \subseteq (a,b)$.

Let's first show that $\int_E f = 0$ for all measurable sets $E \subseteq (a,b)$.

This is true for any open sets (because it is the countable disjoint union of open intervals) and G_δ sets (countable intersections of open sets), since any G_δ set can be represented by the intersection of a countable descending collection of open sets.

Every measurable subset E of $[a,b]$ is of the form $G \sim E_0$, where G is a G_δ subset of (a,b) and $m(E_0)=0$.

Thus we have:

$\int_E f + \int_{E_0} f = \int_G f = 0$ and $\int_{E_0} f = 0$, since $m(E_0)=0$.

Thus $\int_E f = 0$ for all measurable sets $E \subseteq (a,b)$.

Now define $E^+ = \{x \in [a,b] \,|\, f(x) \geq 0\}$, $\quad E^- = \{x \in [a,b] \,|\, f(x) < 0\}$.

$\int_a^b f^+ = \int_{E^+} f = 0$ and $\int_a^b f^- = \int_{E^-} f = 0$.

Thus $f^+ = 0$ a.e. on E^+ and $f^- = 0$ a.e. on E^-.

Hence $f = 0$ a.e. on $[a,b]$.

Theorem: Let f be integrable over the closed, bounded interval $[a, b]$. Then $\frac{d}{dx}\left[\int_a^x f\right] = f(x)$ for almost all $x \in [a, b]$.

Proof: Let $F(x) = \int_a^x f$ for $x \in [a, b]$.

F is absolutely continuous on $[a, b]$.

Thus F is differentiable a.e. on $[a, b]$ and F' is integrable.

To show that $F' - f = 0$ a.e. on $[a, b]$ we just need to show that $\int_{x_1}^{x_2}[F' - f] = 0$ for all $[x_1, x_2] \subseteq (a, b)$ (by the previous lemma).

Since F is absolutely continuous we know $\int_{x_1}^{x_2} F' = F(x_2) - F(x_1)$.

So

$$\int_{x_1}^{x_2}[F' - f] = \int_{x_1}^{x_2} F' - \int_{x_1}^{x_2} f = F(x_2) - F(x_1) - \int_{x_1}^{x_2} f$$

$$= \int_a^{x_2} f - \int_a^{x_1} f - \int_{x_1}^{x_2} f = 0.$$

So $\frac{d}{dx}\left[\int_a^x f\right] = f(x)$ for almost all $x \in [a, b]$.

Ex. Let f be of bounded variation on $[a,b]$ and define $v(x) = TV\left(f_{[a,x]}\right)$ for all $x \in [a,b]$. Show that $|f'| \leq v'$ a.e. on $[a,b]$ and $\int_a^b |f'| \leq TV(f)$. Show this is an equality if f is absolutely continuous.

Take a partition $P = \{x_1, x_2\}$, $x_1, x_2 \in [a,b]$. Then

$$|f(x_2) - f(x_1)| = V(f,P) \leq TV(f_{[x_1,x_2]})$$
$$= TV\left(f_{[a,x_2]}\right) - TV(f_{[a,x_1]}).$$

So we have:

$$\frac{|f(x_2)-f(x_1)|}{x_2-x_1} \leq \frac{TV\left(f_{[a,x_2]}\right)-TV\left(f_{[a,x_1]}\right)}{x_2-x_1} = \frac{v(x_2)-v(x_1)}{x_2-x_1}.$$

Since f is of bounded variation, f' exists a.e. on $[a,b]$. It also means that $TV\left(f_{[a,x]}\right) = v(x)$ is absolutely continuous and so v' exists a.e. on $[a,b]$.

Thus

$$\lim_{x_2 \to x_1} \frac{|f(x_2)-f(x_1)|}{x_2-x_1} \leq \lim_{x_2 \to x_1} \frac{v(x_2)-v(x_1)}{x_2-x_1}.$$

So where these limits exist (a.e. on $[a,b]$) and

$$|f'| \leq v'.$$

Thus we have: $\int_a^b |f'| \leq \int_a^b v' = v(b) - v(a)$ because v is absolutely continuous.

Now $v(b) - v(a) = TV(f)$ on $[a, b]$ so

$$\int_a^b |f'| \leq TV(f).$$

Now if f is absolutely continuous on $[a, b]$ then

$$\int_{x_1}^{x_2} f' = f(x_2) - f(x_1) \quad \text{for } (x_1, x_2) \subseteq (a, b).$$

f has bounded variation so for any partition P of $[a, b]$

$$V(f, P) = \sum_{k=1}^{n} |f(b_k) - f(a_k)| = \sum_{k=1}^{n} \left| \int_{a_k}^{b_k} f' \right|$$

$$\leq \sum_{k=1}^{n} \int_{a_k}^{b_k} |f'| = \int_a^b |f'|.$$

Thus we have: $TV(f) \leq \int_a^b |f'|$.

However, in the first part we showed $TV(f) \geq \int_a^b |f'|$.

Thus if f is absolutely continuous $TV(f) = \int_a^b |f'|$.