Absolutely Continuous Functions

Def. A real valued function f on a closed interval $[a, b]$ is said to be **absolutely continuous** on $[a, b]$ if for each $\epsilon > 0$, there is a $\delta > 0$ such that for every disjoint collection $\{(a_k, b_k)\}_{k=1}^{n}$ of open intervals in (a, b) if $\sum_{k=1}^{n} |b_k - a_k| < \delta$ then $\sum_{k=1}^{n} |f(b_k) - f(a_k)| < \epsilon$.

Notice that if the finite collection is a single set we get the definition for uniform continuity. Thus absolutely continuous implies uniformly continuous (but not the other way around).

Ex. The Cantor function φ is increasing and continuous on $[0,1]$ (and hence uniformly continuous), but it is not absolutely continuous.

In the n^{th} stage of construction the Cantor set is a disjoint collection $\{[c_k, d_k]\}_{k=1}^{k=2^n}$ of 2^n subintervals of $[0,1]$ each of length 3^{-n}.

For example $A_2 = \left[0, \frac{1}{9}\right] \cup \left[\frac{2}{9}, \frac{1}{3}\right] \cup \left[\frac{2}{3}, \frac{7}{9}\right] \cup [\frac{8}{9}, 1]$.

φ is constant on each of the intervals that comprise the complement in $[0,1]$ of this collection of intervals.

Since φ is increasing and $\varphi(1) - \varphi(0) = 1$;

$$\sum_{k=1}^{k=2^n} |d_k - c_k| = \left(\frac{2}{3}\right)^n \quad \text{while} \quad \sum_{k=1}^{k=2^n} |\varphi(d_k) - \varphi(c_k)| = 1.$$

(Since φ takes on the values $\{\frac{1}{2^n}, \frac{2}{2^n}, \frac{3}{2^n}, \dots, \frac{2^n - 1}{2^n}\}$ on the $2^n - 1$ open intervals).

But if $\epsilon = 1$ there is no $\delta > 0$ where if $\sum_{k=1}^{k=2^n} |d_k - c_k| < \delta$ then $\sum_{k=1}^{k=2^n} |f(d_k) - f(c_k)| < \epsilon$.

It's not hard to show that linear combinations of absolutely continuous functions are also absolutely continuous, however, compositions of absolutely continuous functions need not be absolutely continuous.

Prop. If f is Lipschitz on a closed, bounded interval $[a, b]$ then it is absolutely continuous.

Proof: Let $c > 0$ be a Lipschitz constant for f on $[a, b]$. So

$$|f(u) - f(v)| \leq c|u - v| \quad \text{for all } u, v \in [a, b].$$

If we just take $\delta = \frac{\epsilon}{c}$ then

$$\sum_{k=1}^{n} |b_k - a_k| < \delta = \frac{\epsilon}{c} \Longrightarrow c \sum_{k=1}^{n} |b_k - a_k| < \epsilon.$$

But since f is Lipschitz with constant c:

$$\sum_{k=1}^{n} |f(b_k) - f(a_k)| \leq c \sum_{k=1}^{n} |b_k - a_k| < \epsilon.$$

Hence f is absolutely continuous on $[a, b]$.

Note: there are functions that are absolutely continuous but are not Lipschitz. For example $f(x) = \sqrt{x}$ for $0 \leq x \leq 1$.

Theorem: Let the function f be absolutely continuous on the closed, bounded interval $[a, b]$. Then f is the difference of increasing absolutely continuous functions and hence of bounded variation.

Proof: First let's show that f is of bounded variation.

Let δ correspond to $\epsilon = 1$.

Let P be a partition of $[a, b]$ into N closed intervals $\{[c_k, d_k]\}_{k=1}^{N}$ each of length less than δ.

Since $|d_k - c_k| < \delta$ for each $[c_k, d_k]$, any partition $\{\beta_0, \beta_1, \dots, \beta_m\}$ of $[c_k, d_k]$ will have: $\sum_{j=1}^{m} |f(\beta_j) - f(\beta_{j-1})| < \epsilon = 1$, that is $TV\left(f_{[c_k, d_k]}\right) < 1$ for $1 \leq k \leq N$.

By the additivity of the total variation of disjoint intervals:

$$TV(f) = \sum_{k=1}^{N} TV\left(f_{[c_k, d_k]}\right) < N.$$

So f is of bounded variation.

Since f is of bounded variation we can write:

$$f(x) = \left[f(x) + TV\left(f_{[a,x]}\right)\right] - TV(f_{[a,x]}).$$

To show that f is the difference of absolutely continuous functions we just need to show that $TV(f_{[a,x]})$ is absolutely continuous.

Since f is absolutely continuous given any $\epsilon > 0$ choose $\delta > 0$ such that if $\sum_{k=1}^{n}|d_k - c_k| < \delta$ then $\sum_{k=1}^{n}|f(d_k) - f(c_k)| < \frac{\epsilon}{2}$.

Let P_k be a partition of $[c_k, d_k]$ for $1 \le k \le n$.

Since $\sum_{k=1}^{n}|d_k - c_k| < \delta$ we have: $\sum_{k=1}^{n} V\left(f_{[c_k,d_k]},\ P_k\right) < \frac{\epsilon}{2}$.

Now taking the supremum:

$$\sum_{k=1}^{n} TV\left(f_{[c_k,d_k]}\right) \le \frac{\epsilon}{2} < \epsilon.$$

Since $TV\left(f_{[a,v]}\right) - TV\left(f_{[a,u]}\right) = TV\left(f_{[u,v]}\right)$ for $a \le u < v \le b$

$$TV\left(f_{[a,d_k]}\right) - TV\left(f_{[a,c_k]}\right) = TV\left(f_{[c_k,d_k]}\right).$$

Hence if $\sum_{k=1}^{n}|d_k - c_k| < \delta$

$$\sum_{k=1}^{n} \left| TV\left(f_{[a,d_k]}\right) - TV\left(f_{[a,c_k]}\right)\right| < \epsilon.$$

So $TV(f_{[a,x]})$ is absolutely continuous.

Note: Lipschitz$\Longrightarrow$ absolutely continuous$\Longrightarrow$ bounded variation.

Ex. $f(x) = \sqrt{x}$ is absolutely continuous on $[0,1]$, but not Lipschitz since $f'(x)$ is not bounded on $[0,1]$.

Ex. $\varphi(x)$, the Cantor function is not absolutely continuous on $[0,1]$ but has bounded variation since it's an increasing function.

Ex. Prove that $f(x) = x^2$ is absolutely continuous on $[-1,1]$.

$f'(x) = 2x$ so on $[-1,1]$, $|f'(x)| \leq 2$. Thus $f(x)$ is Lipschitz (with $c = 2$) and thus absolutely continuous.

Ex. Prove $f(x) = x\cos\left(\frac{\pi}{2x}\right)$ if $0 < x \leq 1$

$= 0$ if $x = 0$

is not absolutely continuous on $[0,1]$.

As we saw earlier, $f(x)$ is not of bounded variation on $[0,1]$, thus it can't be absolutely continuous.

Theorem: Let f be continuous on the closed, bounded interval $[a, b]$. Then f is absolutely continuous on $[a, b]$ if and only if the family of divided difference functions $\{Diff_h f\}_{0<h\leq 1}$ is uniformly integrable over $[a, b]$.

Proof: We will prove that if f is absolutely continuous on $[a, b]$ then $\{Diff_h f\}_{0<h\leq 1}$ is uniformly integrable over $[a, b]$ (this is the statement that we will use later).

From the preceding theorem we know that since f is absolutely continuous on $[a, b]$ it can be written as the difference of two increasing absolutely continuous functions.

Thus we can assume that f is increasing.

To prove uniform integrability of $\{Diff_h f\}_{0<h\leq 1}$ we must show given any $\epsilon > 0$ there is a $\delta > 0$ such that for each measurable subset E of (a, b)

$$\int_E Diff_h f < \epsilon \text{ if } m(E) < \delta \text{ and } 0 \leq h \leq 1.$$

Earlier we had a theorem that said given a measurable set E there exists a G_δ set G with $E \subseteq G$ and $m(G \sim E) = 0$.

But every G_δ is the intersection of a descending sequence of open sets.

In addition, every open set is the disjoint union of a countable collection of open intervals.

Therefore every open set is the union of an ascending sequence of open sets, each of which is the union of a finite disjoint collection of open intervals.

Thus by the continuity of integration we just need to show:

$$\int_E Diff_h f < \epsilon \text{ if } m(E) < \delta \text{ and } 0 \leq h \leq 1, \text{ where } E = (\cup_{k=1}^{n}(c_k, d_k).$$

Choose $\delta > 0$ such that:

$$\sum_{k=1}^{n}|d_k - c_k| < \delta \text{ implies } \sum_{k=1}^{n}|f(d_k) - f(c_k)| < \frac{\epsilon}{2}.$$

$$\int_u^v Diff_h f = \int_u^v \frac{f(x+h)-f(x)}{h}$$

$$= \frac{1}{h}[\int_u^v f(x+h) - \int_u^v f(x)]$$

Let $w = x + h$

$$= \frac{1}{h}[\int_{u+h}^{v+h} f(w) - \int_u^v f(x)]$$

$$= \frac{1}{h}\left[\int_v^{v+h} f - \int_u^{u+h} f\right]$$

Let $t = x - v$;   Let $t = x - u$

$$= \frac{1}{h}\left[\int_0^h (f(t+v) - f(t+u))\right]$$

$$= \frac{1}{h}\left[\int_0^h g(t)\right]; \qquad g(t) = f(t+v) - f(t+u).$$

If $\{[c_k, d_k]\}_{k=1}^n$ is disjoint then:

$\int_E \; Diff_h f = \frac{1}{h}\int_0^h g(t)$; where $E = \bigcup_{k=1}^n (c_k, d_k)$

and $g(t) = \sum_{k=1}^n |f(d_k + t) - f(c_k + t)|$, for $0 \le t \le 1$.

If $\sum_{k=1}^n |d_k - c_k| < \delta$ then for $0 \le t \le 1$, $\sum_{k=1}^n |(d_k + t) - (c_k + t)| < \delta$ and therefore $g(t) < \frac{\epsilon}{2}$.

Thus: $\int_E \; Diff_h f = \frac{1}{h}\int_0^h g(t) < \frac{\epsilon}{2}$.

Hence $\int_E \; Diff_h f < \frac{\epsilon}{2}$ if $m(E) < \delta$,

where $E = \bigcup_{k=1}^n (c_k, d_k)$ and $0 \le h \le 1$.