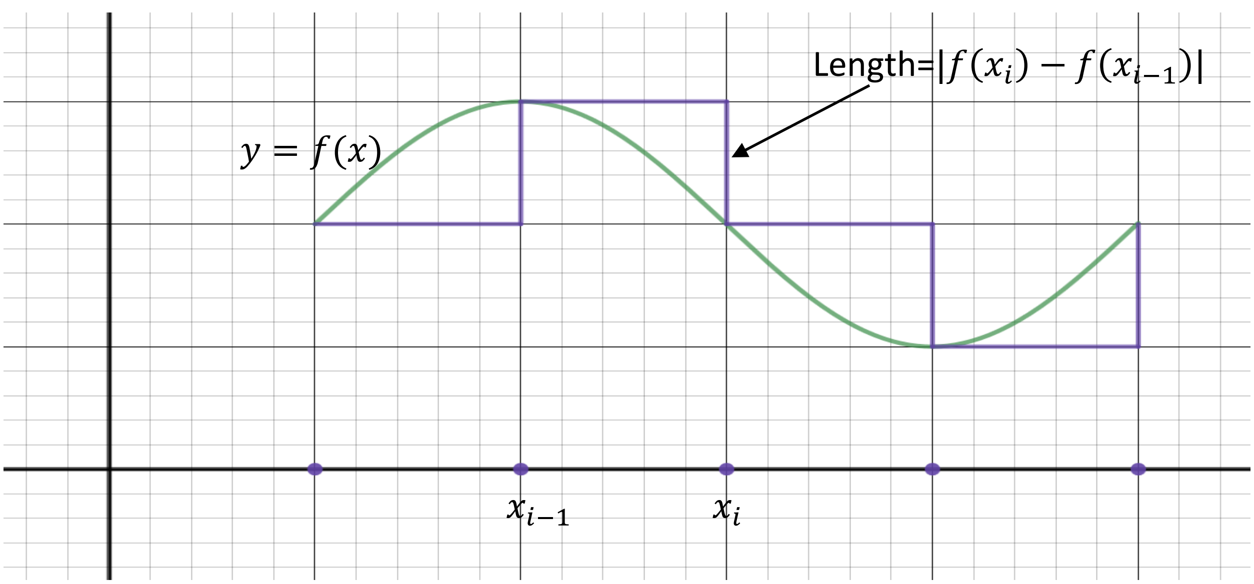
Functions of Bounded Variation

By Lebesgue's theorem we know that a monotonic function on an open interval is differentiable a.e.. Hence a function that is the difference of two increasing (or decreasing) functions is also differentiable a.e.. We now want to characterize the class of functions on a closed, bounded interval which are the difference of two increasing (or decreasing) functions.

Def. Let f be a real valued function defined on a closed, bounded interval $[a, b]$ and P a partition $\{x_0, x_1, x_2, \dots, x_k\}$ of $[a, b]$. The **variation of f with respect to P** is defined as:

$$V(f, P) = \sum_{i=1}^{k} |f(x_i) - f(x_{i-1})|.$$



The **total variation of f** on $[a, b]$ is defined as:

$$TV(f) = \sup\{V(f, P) |\ P \text{ a partition of } [a, b]\}.$$

Def. A real valued function f on the closed, bounded interval $[a, b]$ is said to be of **bounded variation** if $TV(f) < \infty$.

Ex. If f is an increasing function on $[a, b]$, then f is of bounded variation and $TV(f) = f(b) - f(a)$.

Given any partition P of $[a, b]$:

$$V(f, P) = \sum_{i=1}^{k} |f(x_i) - f(x_{i-1})|$$
$$= \sum_{i=1}^{k}(f(x_i) - f(x_{i-1})) = f(b) - f(a).$$

Thus $TV(f) = \sup\limits_{P} V(f, P) = f(b) - f(a)$.

Ex. Let f be a Lipschitz function on $[a, b]$. Then f is of bounded variation on $[a, b]$ and $TV(f) \leq c(b - a)$ where c is the Lipschitz constant, $|f(u) - f(v)| \leq c|u - v|$ for all $u, v \in [a, b]$.

Let $P = \{x_0, x_1, x_2, \dots, x_k\}$ be any partition of $[a, b]$. Then:

$$V(f, P) = \sum_{i=1}^{k} |f(x_i) - f(x_{i-1})| \leq \sum_{i=1}^{k} c|x_i - x_{i-1}| = c|b - a|.$$

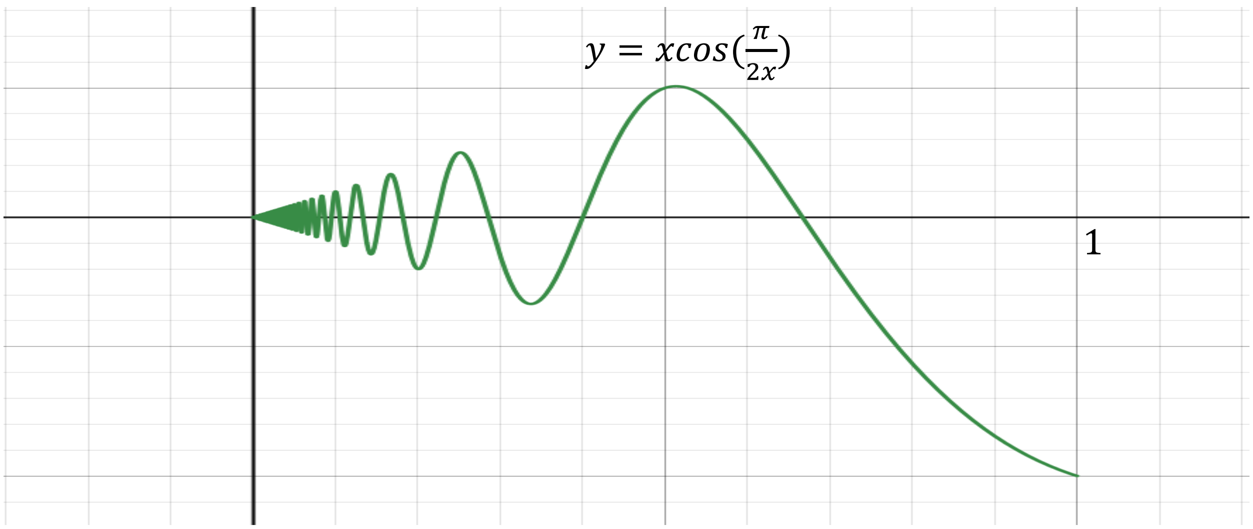
Thus $c|b - a|$ is an upper bound for $V(f, P)$ and $TV(f) \leq c(b - a)$.

Ex. Define f on $[0,1]$ by

$$f(x) = xcos(\frac{\pi}{2x}) \quad \text{if } 0 < x \le 1$$

$$= 0 \quad \text{if } x = 0.$$

f is continuous on $[0,1]$, and therefore bounded, but does not have bounded variation.



If we take the partition: $P_n = \{0, \frac{1}{2n}, \frac{1}{2n-1}, \frac{1}{2n-2}, \dots, \frac{1}{3}, \frac{1}{2}, 1\}$ of $[0,1]$

$$f(x_0) = 0$$

$$f(x_1) = \frac{1}{2n}\cos\left(\frac{\pi}{2\left(\frac{1}{2n}\right)}\right) = \frac{1}{2n}\cos(n\pi) = \pm\frac{1}{2n}$$

$$f(x_2) = \frac{1}{2n-1}\cos\left(\frac{\pi}{2\left(\frac{1}{2n-1}\right)}\right) = \frac{1}{2n-1}\cos\left(\frac{2n-1}{2}\pi\right) = 0$$

$$f(x_3) = \frac{1}{2n-2}\cos\left(\frac{\pi}{2\left(\frac{1}{2n-2}\right)}\right) = \frac{1}{2n-2}\cos\left(\frac{2n-2}{2}\pi\right) = \pm\frac{1}{2n-2}$$

$$f(x_4) = \frac{1}{2n-3}\cos\left(\frac{2n-3}{2}\pi\right) = 0$$

$\vdots$

So $|f(x_1) - f(x_0)| = \frac{1}{2n}$

$|f(x_2) - f(x_1)| = \frac{1}{2n}$

$|f(x_3) - f(x_2)| = \frac{1}{2n-2}$

$|f(x_4) - f(x_3)| = \frac{1}{2n-2}$; etc.

Thus $V(f, P_n) = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}$; which diverges as n goes to ∞.

So f is not of bounded variation.

Ex. Notice that if f is $C^1(a,b)$ (i.e. $f'(x)$ is continuous on (a,b)) and continous on $[a,b]$, then for any partition $P = \{x_0, x_1, x_2, \ldots, x_k\}$:

$$f(x_i) - f(x_{i-1}) = \int_{x_{i-1}}^{x_i} f'$$

by the fundamental theorem of Calculus.

Thus we have:

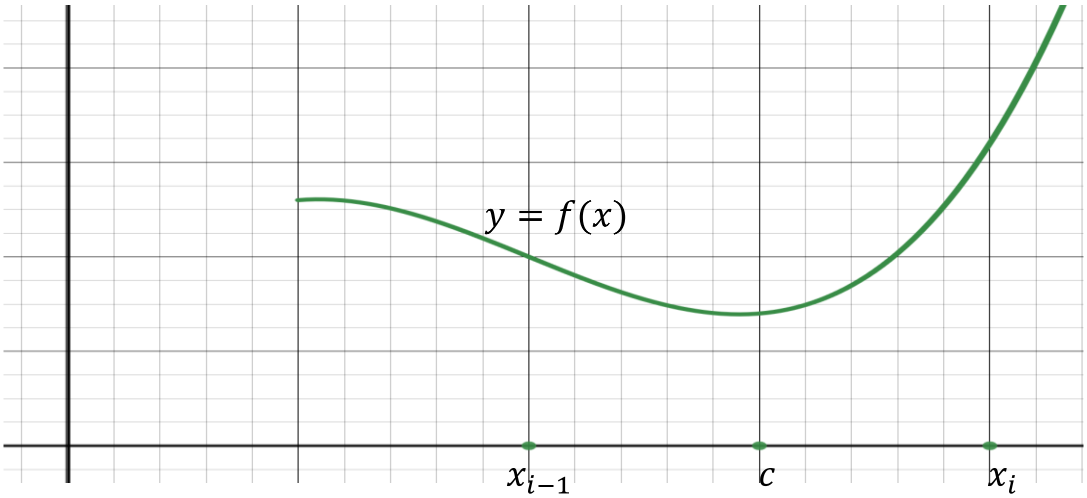
$$|f(x_i) - f(x_{i-1})| = |\int_{x_{i-1}}^{x_i} f'| \leq \int_{x_{i-1}}^{x_i} |f'|.$$

So

$$\sum_{i=1}^{n} |f(x_i) - f(x_{i-1})| \leq \int_a^b |f'|.$$

Thus $TV(f) \leq \int_a^b |f'|$, and f is of bounded variation as long as $\int_a^b |f'| < \infty$.

Notice if $c \in [a, b]$ and c is not one of the endpoints of a partition P, we can create a refinement P' of P by adding c.



Then by the triangle inequality: $V(f, P) \leq V(f, P')$. Here's why.

The triangle inequality says $|a + b| \leq |a| + |b|$ for all $a, b \in \mathbb{R}$.

$$|f(x_i) - f(x_{i-1})| \leq |f(x_i) - f(c)| + |f(c) - f(x_{i-1})|$$

where $a = f(x_i) - f(c), \quad b = f(c) - f(x_{i-1})$

$$a + b = f(x_i) - f(x_{i-1}).$$

Thus $TV(f)$ can be calculated as the supremum of $V(f, P)$ over all partitions containing c. So if a partition P includes c, we can break P up into a partition of $[a, c]$ and $[c, b]$ so that :

$$V\left(f_{[a,b]}, P\right) = V\left(f_{[a,c]}, P\right) + V\left(f_{[c,b]}, P\right).$$

By taking the supremum we get:

$$TV\left(f_{[a,b]}\right) = TV\left(f_{[a,c]}\right) + TV\left(f_{[c,b]}\right).$$

So if f is of bounded variation on $[a,b]$ then:

$$TV\left(f_{[a,v]}\right) = TV\left(f_{[a,u]}\right) + TV\left(f_{[u,v]}\right) \quad \text{for } a \le u < v \le b.$$

Thus we have:

$$TV\left(f_{[a,v]}\right) - TV\left(f_{[a,u]}\right) = TV\left(f_{[u,v]}\right) \ge 0 \quad \text{for } a \le u < v \le b.$$

So $g(x) = TV(f_{[a,x]})$ is an increasing function on $[a,b]$.

g is called the **total variation function for f**.

Notice that if $P = \{u,v\}$ then:

$$f(u) - f(v) \le |f(u) - f(v)| = V\left(f_{[u,v]}, P\right) \le TV\left(f_{[u,v]}\right)$$
$$= TV\left(f_{[a,v]}\right) - TV\left(f_{[a,u]}\right).$$

Thus we have:

$$f(v) + TV\left(f_{[a,v]}\right) \ge f(u) + TV(f_{[a,u]}) \quad \text{for } a \le u < v \le b.$$

So $h(x) = f(x) + TV(f_{[a,x]})$ is an increasing function on $[a,b]$.

Thus we have shown:

Lemma: Let f be of bounded variation on a closed, bounded interval $[a,b]$.
Then f can be written as the difference of two increasing functions on $[a,b]$:
$f(x) = \left[f(x) + TV\left(f_{[a,x]}\right)\right] - TV(f_{[a,x]})$.

Jordan's Theorem: A function f is of bounded variation on a closed, bounded interval $[a, b]$ if and only if it is the difference of two increasing function on $[a, b]$.

Proof: The previous lemma says if f is of bounded variation on $[a, b]$ then it can be written as the difference of two increasing functions on $[a, b]$.

Now suppose $f = g - h$ on $[a, b]$, where g, h are increasing on $[a. b]$.

Let's show that f is of bounded variation.

Let P be a partition $\{x_0, x_1, x_2, \dots, x_k\}$ of $[a, b]$.

$$
\begin{aligned}
V(f, P) &= \sum_{i=1}^{k} |f(x_i) - f(x_{i-1})| \\
&= \sum_{i=1}^{k} \big|\big(g(x_i) - g(x_{i-1})\big) + \big(h(x_{i-1}) - h(x_i)\big)\big| \\
&\leq \sum_{i=1}^{k} \big|\big(g(x_i) - g(x_{i-1})\big)\big| + \sum_{i=1}^{k} \big|\big(h(x_{i-1}) - h(x_i)\big)\big| \\
&= |g(b) - g(a)| + |h(b) - h(a)| < \infty.
\end{aligned}
$$

So $TV(f) < \infty$.

We call $f(x) = \left[f(x) + TV\big(f_{[a,x]}\big)\right] - TV\big(f_{[a,x]}\big)$ the **Jordan Decomposition** of f.

Corollary: If f is a function of bounded variation on a closed, bounded interval $[a, b]$, then it is differentiable a.e. on (a, b) and f' is integrable over $[a, b]$.

Proof: Since f is the difference of two increasing function on $[a, b]$, it is differentiable a.e. by Lebesgue's theorem.

A corollary to Lebesgue's theorem is that if f is increasing on $[a, b]$ then f' is integrable over $[a, b]$.

Thus if $f = g - h,\ \ g, h$ increasing then g', h' are integrable over $[a. b]$ and hence so is f'.