Differentiability of Monotonic Functions

Def. A closed interval $[c, d]$ is said to be **nondegenerate** if $c < d$.

Def. A collection F of closed, bounded, nondegenerate intervals is called **a Vitali covering of a set $\boldsymbol{E}$** if for each point $x \in E$ and $\epsilon > 0$, there is an interval $I \in F$ that contains x and has $l(I) < \epsilon$.

Ex. A Vitali covering of $[0,1]$.

Let $R = \{r_1, r_2, r_3, \dots\}$ be the set of rational numbers in $[0,1]$.

Let $I_{k,n} = [r_k - \frac{1}{n}, r_k + \frac{1}{n}]$.

Then $F = \{I_{k,n}\}_{k,n=1}^{k,n=\infty}$, is a Vitali covering of $[0,1]$.

The Vitali Covering Lemma: Let E be a set of finite outer measure and F a collection of closed, bounded, nondegenerate intervals that is a Vitali covering of E. Then for each $\epsilon > 0$ there is a finite disjoint subcollection $\{I_k\}_{k=1}^{n}$ of F for which $m^*(E \sim \bigcup_{k=1}^{n} I_k) < \epsilon$.

Let x be a point of the domain of a real valued function f.

Def. We define the **upper and lower derivative of f** at x by

$$\overline{D}f(x) = \lim_{h\to 0}[\sup_{0<|t|\le h} \frac{f(x+t)-f(x)}{t}]$$

$$\underline{D}\, f(x) = \lim_{h\to 0}[\inf_{0<|t|\le h} \frac{f(x+t)-f(x)}{t}].$$

Notice that $\overline{D}f(x) \geq \underline{D}f(x)$.

If $\overline{D}f(x) = \underline{D}f(x)$ and is finite, we say **f is differentiable at x** and define $f'(x)$ to be the common value.

Ex. Let $f(x) = 0$ if $x \in \mathbb{Q}$

$= 1$ if $x \notin \mathbb{Q}$.

$$\overline{D}f(0) = \lim_{h\to 0}[\sup_{0<|t|\le h} \frac{f(0+t)-f(0)}{t}] = \lim_{h\to 0}\left[\sup_{0<|t|\le h} \frac{f(t)}{t}\right] = \infty$$

$$\underline{D}f(0) = \lim_{h\to 0}[\inf_{0<|t|\le h} \frac{f(0+t)-f(0)}{t}] = \lim_{h\to 0}\left[\inf_{0<|t|\le h} \frac{f(t)}{t}\right] = -\infty.$$

Since $\overline{D}f(0) \neq \underline{D}f(0)$, $f(x)$ does not have a derivative at $x = 0$.

If $f(x)$ is continuous on $[a, b]$, and differentiable on (a, b), the Mean Value Theorem tells us that there is a $c \in (a, b)$ such that

$$\frac{f(b)-f(a)}{b-a} = f'(c).$$

If we know that $f'(x) \geq \alpha$ for all $a < x < b$ then

$$\frac{f(b)-f(a)}{b-a} = f'(c) \geq \alpha \quad \text{or} \quad f(b) - f(a) \geq \alpha(b - a).$$

We have the following generalization for increasing functions.

Lemma: Let f be an increasing function on the closed, bounded interval $[a, b]$. Then for each $\alpha > 0$

$$m^*\{x \in (a, b) |\ \overline{D}f(x) \geq \alpha\} \leq \frac{1}{\alpha}[f(b) - f(a)]$$

and $$m^*\{x \in (a, b) |\ \overline{D}f(x) = \infty\} = 0.$$

Proof: Let $\alpha > 0$.

Define $E_\alpha = \{x \in (a, b) |\ \overline{D}f(x) \geq \alpha\}$.

Choose $\alpha' \in (0, \alpha)$.

Let F be a collection of closed, bounded intervals $[c, d]$ contained in (a, b) for which: $f(d) - f(c) \geq \alpha'(d - c)$.

Since $\overline{D}f(x) \geq \alpha$ on E_α, F is a Vitali covering of E_α.

By the Vitali covering lemma there is a finite, disjoint subcollection $\{[c_k, d_k]\}_{k=1}^n$ of F for which

$$m^*(E_\alpha \sim \cup_{k=1}^n [c_k, d_k]\} < \epsilon.$$

Since $E_\alpha \subseteq (\cup_{k=1}^n [c_k, d_k]) \cup (E_\alpha \sim \cup_{k=1}^n [c_k, d_k])$ we have

$$m^*(E_\alpha) \leq \sum_{k=1}^n m^*([c_k, d_k]) + m^*(E_\alpha \sim \cup_{k=1}^n [c_k, d_k])$$

$$< \sum_{k=1}^n (d_k - c_k) + \epsilon.$$

But F is the set of $[c,d]$ with $f(d)-f(c)\geq\alpha'(d-c)$. So

$$m^*(E_\alpha)\leq\frac{1}{\alpha'}\left[\sum_{k=1}^{n}(f(d_k)-f(c_k))\right]+\epsilon.$$

However, f is increasing on $[a,b]$ and $\{[c_k,d_k]\}_{k=1}^{n}$ are disjoint so

$$\sum_{k=1}^{n}(f(d_k)-f(c_k))\leq f(b)-f(a).$$

Thus for each $\epsilon>0$ and $\alpha'\in(0,\alpha)$

$$m^*(E_\alpha)\leq\frac{1}{\alpha'}[f(b)-f(a)]+\epsilon.$$

Hence $m^*\{x\in(a,b)|\,\overline{D}f(x)\geq\alpha\}\leq\frac{1}{\alpha}[f(b)-f(a)].$

For each $n\in\mathbb{Z}^+$, $\{x\in(a,b)|\,\overline{D}f(x)=\infty\}\subseteq E_n$; therefore

$$m^*\{x\in(a,b)|\,\overline{D}f(x)=\infty\}\leq m^*(E_n)\leq\frac{1}{n}\big(f(b)-f(a)\big).$$

Thus $m^*\{x\in(a,b)|\,\overline{D}f(x)=\infty\}=0.$

Lebesgue's Theorem: If the function f is monotonic on the open interval (a,b), then it is differentiable almost everywhere on (a,b).

Proof: Assume f is increasing.

Also assume (a,b) is bounded. If it's not, express it as the union of ascending open, bounded intervals and use the continuity of measure.

The set of points where $\overline{D}f(x) > \underline{D}f(x)$ is

$$\bigcup_{\alpha,\beta\in\mathbb{Q}} E_{\alpha,\beta} = \bigcup_{\alpha,\beta\in\mathbb{Q}}\{x \in (a,b) \mid \overline{D}f(x) > \alpha > \beta > \underline{D}f(x)\}\ .$$

By countable subadditivity of outer measure we only need to prove the assertion for each $E_{\alpha,\beta}$.

Fix $\alpha, \beta \in \mathbb{Q}$ with $\alpha > \beta,$ and let $E = E_{\alpha,\beta}$.

Let $\epsilon > 0.$

Choose an open set O for which: $E \subseteq O \subseteq (a,b)$ and $m^*(O) < m^*(E) + \epsilon.$

Let F be the collection of closed intervals $[c,d] \subseteq O$ with

$f(d) - f(c) < \beta(d - c).$

Since $\beta > \underline{D}f(x)$ on E, F is a Vitali covering of E.

The Vitali covering lemma says there is a finite disjoint subcollection $\{[c_k,d_k]\}_{k=1}^{n}$ of F for which

$$m^*(E \sim \bigcup_{k=1}^{n}[c_k,d_k]\} < \epsilon.$$

Since $[c_k,d_k] \subseteq O$ for all $1 \le k \le n$

$$\sum_{k=1}^{n}[f(d_k) - f(c_k)] < \beta\sum_{k=1}^{n}[(d_k) - (c_k)] \le \beta m^*(O)$$
$$\le \beta(m^*(E) + \epsilon).$$

From the preceding lemma applied to $[c_k,d_k]$:

$$m^*(E \cap (c_k,d_k)) \le \frac{1}{\alpha}[f(d_k) - f(c_k)].$$

Since $m^*(E \sim \bigcup_{k=1}^n [c_k, d_k]\} < \epsilon$, and

$$E = E \cap (\bigcup_{k=1}^n [c_k, d_k]) \cup (E \sim \bigcup_{k=1}^n [c_k, d_k]), \quad \text{we have:}$$

$$m^*(E) < \sum_{k=1}^n m^*\big(E \cap (c_k, d_k)\big) + \epsilon \le \frac{1}{\alpha} \sum_{k=1}^n [f(d_k) - f(c_k)] + \epsilon.$$

Now since $\sum_{k=1}^n [f(d_k) - f(c_k)] \le \beta(m^*(E) + \epsilon)$ we have:

$$m^*(E) \le \frac{1}{\alpha} \sum_{k=1}^n [f(d_k) - f(c_k)] + \epsilon$$

$$\le \frac{\beta}{\alpha} m^*(E) + \frac{\epsilon}{\alpha} + \epsilon \quad \text{for all } \epsilon > 0.$$

Therefore since $0 \le m^*(E) < \infty$ and $\frac{\beta}{\alpha} < 1$

$$m^*(E) = 0.$$

Let f be integrable over the closed, bounded interval $[a, b]$. Extend f to take on the value $f(b)$ on $(b, b+1]$.

For $0 < h \le 1$ define the divided difference function, $Diff_h f$ and the average value function $Av_h f$ on $[a, b]$ by:

$$\boldsymbol{Diff_h f(x) = \frac{f(x+h) - f(x)}{h}} \quad \text{and} \quad \boldsymbol{Av_h f(x) = \frac{1}{h} \int_x^{x+h} f}.$$

(Recall from first year calculus that the average value of a function $y = f(x)$ over an interval $[a, b]$ is given by: $f_{ave} = \frac{1}{b-a} \int_a^b f(x)dx$).

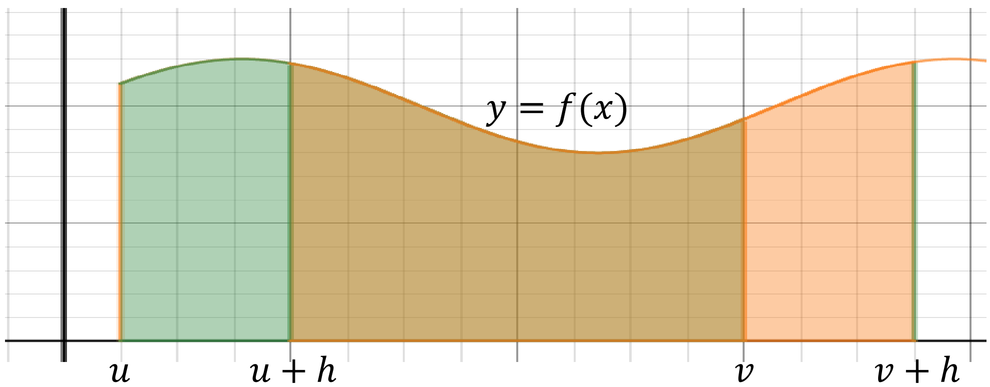
Notice that for $a \leq u < v \leq b$:

$$\int_u^v Diff_h f = \int_u^v \frac{f(x+h)-f(x)}{h}$$

$$= \frac{1}{h}[\int_u^v f(x+h) - \int_u^v f(x)]$$

Now let $w = x + h$

$$= \frac{1}{h}[\int_{w=u+h}^{w=v+h} f(w) - \int_{x=u}^{x=v} f(x)]$$



$$= \frac{1}{h}[\int_v^{v+h} f - \int_u^{u+h} f]$$

$$= Av_h f(v) - Av_h f(u).$$

This looks a lot like $\int_u^v f'(x)dx = f(v) - f(u)$, the fundamental theorem of Calculus.

Corollary: Let f be an increasing function on the closed, bounded interval $[a, b]$. Then $f'(x)$ is integrable over $[a, b]$ and

$$\int_a^b f' \leq f(b) - f(a).$$

Proof: We can extend f to be increasing on $[a, b+1]$ by $f(x) = f(b)$ for $b < x \leq b+1$.

Since f is increasing, it is measurable and therefore $\frac{f(x+h)-f(x)}{h}$ is measurable.

Lebesgue's theorem says f' exists a.e. on (a, b), thus $\{Diff_{\frac{1}{n}} f(x)\}$ is a sequence of nonnegative measurable functions that converges pointwise a.e. on $[a, b]$ to f'.

By Fatou's lemma:

$$\int_a^b f' \leq liminf \int_a^b Diff_{\frac{1}{n}} f(x) .$$

Since $\int_u^v Diff_h f = Av_h f(v) - Av_h f(u)$ we have:

$$\int_a^b Diff_{\frac{1}{n}} f(x) = Av_{\frac{1}{n}} f(b) - Av_{\frac{1}{n}} f(a)$$

$$= \frac{1}{1/n} \int_b^{b+\frac{1}{n}} f - \frac{1}{1/n} \int_a^{a+\frac{1}{n}} f$$

$$= f(b) - \frac{1}{1/n} \int_a^{a+\frac{1}{n}} f \text{ since } f(x) = f(b) \text{ for } b < x \leq b+1.$$

So: $\int_a^b Diff_{\frac{1}{n}} f(x) = f(b) - \frac{1}{1/n} \int_a^{a+\frac{1}{n}} f \leq f(b) - f(a).$

since f is increasing.

Thus we have: $$\lim_{n\to\infty} \sup\left[\int_a^b Diff_{\frac{1}{n}} f(x)\right] \le f(b) - f(a).$$

Hence:

$$\int_a^b f' \le \lim_{n\to\infty} \inf\left[\int_a^b Diff_{\frac{1}{n}} f(x)\right]$$
$$\le \lim_{n\to\infty} \sup\left[\int_a^b Diff_{\frac{1}{n}} f(x)\right] \le f(b) - f(a).$$

Ex. The Cantor function, φ, is increasing and continuous on $[0,1]$. It also has the property that $\varphi(1) = 1$, $\varphi(0) = 0$, and $\varphi'(x) = 0$ a.e. on $[0,1]$.

Thus $\int_0^1 \varphi'(x) = 0$, but $\varphi(1) - \varphi(0) = 1$, so

$$\int_0^1 \varphi'(x) < \varphi(1) - \varphi(0).$$

Ex. Notice that the corollary to Lebesgue's theorem says that if f is increasing on $[a, b]$ then f' is integrable on $[a, b]$. If f is not increasing on $[a, b]$, f' may not be integrable even if f is continuous on $[a, b]$ and differentiable at every point but 1. For example:

$$f(x) = x^2 sin\frac{1}{x^2} \qquad 0 < x \le 1$$
$$= 0 \qquad\qquad x = 0\ .$$

Has a derivative everywhere but $x = 0$, however, $\int_0^1 |f'|$ is not finite so f' is not Lebesgue integrable.