The Vitali Convergence Theorem

Lemma: Let E be a set of finite measure and $\delta > 0$. Then E is the disjoint union of a finite collection of sets, each of which has measure less than δ.

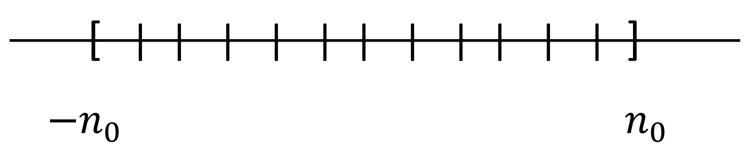
Proof: By the continuity of measure,

$$\lim_{n\to\infty} m(E \sim [-n, n]) = m(\phi) = 0$$

since if $E \sim [-n, n] = E \cap [-n, n]^c = E_n$
then $E_{n+1} \subseteq E_n$ and $m(E) < \infty$.

Choose $n_0 \in \mathbb{Z}^+$ for which, $m(E \sim [-n_0, n_0]) < \delta$.

Now choose a partition of $[-n_0, n_0]$ fine enough so $E \cap [-n_0, n_0]$ is the disjoint union of a finite collection of sets, each of which has measure less than δ.



$$-n_0 = a_0 < a_1 < \cdots < a_m = n_0.$$

Let $E_j = E \cap \left(a_{j-1}, a_j\right]$;   with $m\left(E_j\right) < \delta$.

Then $E_1, \ldots, E_m,\ E \sim [-n_0, n_0]$ works.

Prop. Let f be a measurable function on E. If f is integrable over E, then for $\epsilon > 0$, there is a $\delta > 0$ for which

If $A \subseteq E$ is measurable and $m(A) < \delta$ then $\int_A |f| < \epsilon$.

Conversely, in the case $m(E) < \infty$, if for each $\epsilon > 0$, there is a $\delta > 0$ for which if $A \subseteq E$ is measurable and $m(A) < \delta$, then $\int_A |f| < \epsilon$ then f is integrable over E.

Proof: Let $f = f^+ - f^-$; and establish for f^+ and f^-, so assume f is non-negative.

Assume f is integrable over E. Let $\epsilon > 0$.

By additivity over subdomains, for $a > 0$ we have:

$$\int_A f = \int_{\{x \in A | f(x) < a\}} f + \int_{\{x \in A | f(x) \geq a\}} f$$

$$\leq a \cdot m(A) + \int_{\{x \in A | f(x) \geq a\}} f.$$

Since $\infty > \int_E f \geq \int_A f$, we can choose a large enough that:

$$\int_{\{x \in A | f(x) \geq a\}} f < \frac{\epsilon}{2}$$

So, $\int_A f < a \cdot m(A) + \frac{\epsilon}{2}$.

Choose $\delta = \frac{\epsilon}{2a}$, then if $A \subseteq E$ and $m(A) < \delta = \frac{\epsilon}{2a}$,

then $\int_A f < a\left(\frac{\epsilon}{2a}\right) + \frac{\epsilon}{2} = \epsilon$.

Conversely, suppose $m(E) < \infty$ and that for each $\epsilon > 0$, there is a $\delta > 0$ for which, if $A \subseteq E$ is measurable and $m(A) < \delta$, then

$$\int_A |f| < \epsilon .$$

Let δ correspond to $\epsilon = 1$.

Since $m(E) < \infty$ according to the previous lemma we can write $E = \bigcup_{k=1}^{N} E_k$, disjoint sets of measure less than δ.

Therefore, $\sum_{k=1}^{N} \int_{E_k} f < N$ since $\int_{E_k} f < 1$ for each k.

Thus $\int_E f = \sum_{k=1}^{N} \int_{E_k} f < N$ and f is integrable.

Note: if $m(E) = \infty$, then $f(x) = 1$ has $\int_A |f| < \epsilon$ if $\delta = \epsilon$, but f is not integrable over E.

Def. A family F of measurable functions on E is said to be **uniformly integrable** over E if for each $\epsilon > 0$, there is a $\delta > 0$ such that for each $f \in F$, if $A \subseteq E$ is measurable and $m(A) < \delta$, then $\int_A |f| < \epsilon$.

Ex. Let g be a non-negative integrable function over E. Define:

$$F = \{f \mid f \text{ is measurable and } |f| \leq g \text{ on } E\}.$$

Then F is uniformly integrable. This follows from the previous proposition and that for any measurable subset $A \subseteq E$:

$$\int_A |f| \leq \int_A g$$

Ex. Let $f_n(x) = n$ if $0 \leq x \leq \frac{1}{n}$

$= 0$ if $\frac{1}{n} \leq x \leq 1.$

$\{f_n\}$ are not uniformly integrable over $[0, 1]$ since if $\epsilon = \frac{1}{2}$ then given any $\delta > 0$ there exists an $n \in \mathbb{Z}^+$ such that $\frac{1}{n} < \delta$ and $\int_{[0,\frac{1}{n}]} f_n = 1.$

Prop. Let $\{f_k\}_{k=1}^{n}$ be a finite collection of functions, each of which is integrable over E. Then $\{f_k\}_{k=1}^{n}$ is uniformly integrable.

Proof: Let $\epsilon > 0.$

For $1 \leq k \leq n$ by the previous proposition there is a δ_k such that if $A \subseteq E$ is measurable and $m(A) < \delta_k$, then $\int_A |f_k| < \epsilon.$

Let $\delta = \min\{\delta_1, \dots, \delta_n\}$ then if $A \subseteq E$ is measurable and $m(A) < \delta$, then $\int_A |f_k| < \epsilon$ for all $1 \leq k \leq n.$

Prop. Assume E has finite measure. Let $\{f_n\}$ be uniformly integrable over E. If $\{f_n\} \to f$ pointwise a.e. on E, then f is integrable over E.

Proof: Let $\epsilon = 1$ and $\delta_0 > 0$ such that if $m(A) < \delta_0$ then

$$\int_A |f_n| < \epsilon = 1.$$

Since $m(E) < \infty$ we can express E as the disjoint union of a finite collection of measurable sets $\{E_k\}_{k=1}^N$ such that $m(E_k) < \delta_0$ for $1 \leq k \leq N$.

Thus, $\int_E |f_n| = \sum_{k=1}^N \int_{E_k} |f_n| < N$.

But by Fatou's lemma, $\int_E |f| \leq liminf \int_E |f_n| < N$.

So , f is integrable over E.

Vitali Convergence Theorem: Let E be of finite measure. Suppose $\{f_n\}$ is uniformly integrable over E. If $f_n \to f$ pointwise a.e., then f is integrable over E and $\lim_{n\to\infty} \int_E f_n = \int_E f$.

Proof: The previous proposition tells us f is integrable and hence is finite a.e.

We can assume $f_n \to f$ pointwise by excising the set of measure 0 where it doesn't.

Notice for any measurable subset $A \subseteq E$,

$$\left|\int_E f_n - \int_E f\right| = \left|\int_E (f_n - f)\right|$$

$$\leq \int_E |f_n - f|$$

$$= \int_{E\sim A} |f_n - f| + \int_A |f_n - f|$$

$$\left|\int_E f_n - \int_E f\right| \leq \int_{E\sim A} |f_n - f| + \int_A |f_n| + \int_A |f|.$$

$\{f_n\}$ uniformly integrable $\Longrightarrow$ there is a $\delta > 0$ such that if $m(A) < \delta$, then $\int_A |f_n| < \frac{\epsilon}{3}$.

By Fatou's Lemma: $\int_A |f| \leq liminf \int_A |f_n| < \frac{\epsilon}{3}$.

Since f is real valued and $m(E) < \infty$, by Egoroff's Theorem there is a measurable subset $E_0 \subseteq E$ for which $m(E_0) < \delta$ and $f_n \to f$ uniformly on $E\sim E_0$.

Thus there exists N such that if $n \geq N$ then,

$$|f_n - f| < \frac{\epsilon}{3m(E)} \quad \text{on } E\sim E_0.$$

Now take $A = E_0$ and we get:

$$\left|\int_E f_n - \int_E f\right| \leq \left|\int_{E\sim E_0} f_n - f\right| + \int_{E_0} |f_n| + \int_{E_0} |f|$$

$$< \left(\frac{\epsilon}{3m(E)}\right)\left(m(E\sim E_0)\right) + \frac{\epsilon}{3} + \frac{\epsilon}{3} \leq \epsilon\ .$$

So $\lim_{n\to\infty} \int_E f_n = \int_E f$.

Theorem: Let E be of finite measure. Suppose $\{h_n\}$ is a sequence of non-negative integrable functions that converge pointwise a.e. to $h \equiv 0$. Then,

$\lim_{n\to\infty} \int_E h_n = 0$ if, and only if, $\{h_n\}$ is uniformly integrable.

Proof: If $\{h_n\}$ is uniformly integrable then by the Vitali convergence theorem, $\lim_{n\to\infty} \int_E h_n = 0$.

Now suppose $\lim_{n\to\infty} \int_E h_n = 0$.

Let $\epsilon > 0$. We can find N such that if $n \geq N$ then $\int_E h_n < \epsilon$.

Since each $h_n \geq 0$ on E, if $A \subseteq E$ is measurable and $n \geq N$, then $\int_A h_n < \epsilon$.

According to an earlier proposition, a finite collection $\{h_n\}_{n=1}^{N-1}$ of integrable functions is uniformly integrable.

Thus given $\epsilon > 0$ there exists a $\delta > 0$ such that if $m(A) < \delta$ then $\int_A h_n < \epsilon,\ n = 1, \ldots, N-1$.

Thus $\{h_n\}_{n=1}^{\infty}$ are uniformly integrable.