The Lebesgue Integral $\int_E f$: $f \geq 0$

Def. A measurable function with domain E has **finite support** if $m(\{x \in E \mid f(x) \neq 0\}) < \infty$.

Let $E_0 = \{x \in E \mid f(x) \neq 0\}$ with $m(E_0) < \infty$. If f is bounded and measurable on E we can define:

$$\int_E f = \int_{E_0} f.$$

So what do we do if $m(E_0) = \infty$ or $m(E_0) < \infty$ but f is not bounded?

Def. Let f be a nonnegative measurable function on E. Let

$H_E(f) = \{\text{bounded measurable functions of finite support} \mid 0 \leq h \leq f \text{ on } E\}$.

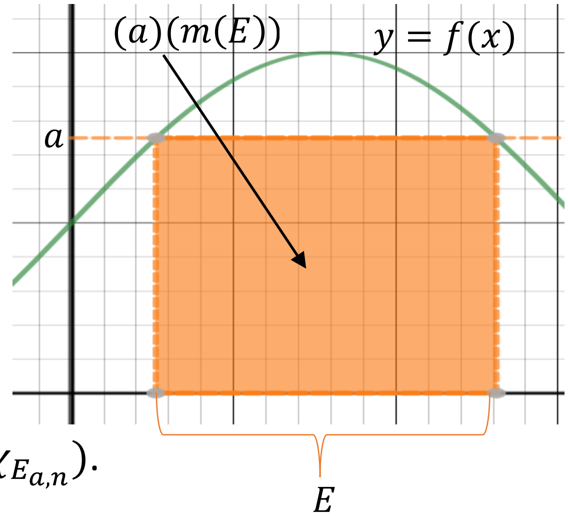
we define $\int_E f$ by

$$\int_E f = \sup\left\{\int_E h \,\middle|\, h \in H_E(f)\right\}.$$

Notice that $\int_E f$ can be finite or ∞.

Chebychev's inequality: Let f be a nonnegative measurable function on E. Then for any $a > 0$,

$$m(\{x \in E \mid f(x) \geq a\}) \leq \frac{1}{a}\int_E f.$$



Proof: Let $E_a = \{x \in E \mid f(x) \geq a\}$.

First let's assume $m(E_a) = \infty$.

Let $E_{a,n} = E_a \cap [-n, n]$ and $\psi_n = a(\chi_{E_{a,n}})$.

ψ_n is a bounded, measurable function of finite support with

$a(m(E_{a,n})) = \int_E \psi_n$ and $0 \leq \psi_n \leq f$ on E for all n.

Notice that: $E_a = \bigcup_{n=1}^{\infty} E_{a,n}$ and $E_{a,n+1} \supseteq E_{a,n}$.

Thus $m(E_a) = \lim_{n\to\infty} m\left(\bigcup_{n=1}^{\infty} E_{a,n}\right) = \lim_{n\to\infty} m(E_{a,n})$.

Thus $\infty = a\big(m(E_a)\big) = a\left(\lim_{n\to\infty} m(E_{a,n})\right) = \lim_{n\to\infty} \int_E \psi_n \leq \int_E f$.

So $m(\{x \in E \mid f(x) \geq a\}) \leq \frac{1}{a}\int_E f$ because both sides are ∞.

Now suppose $m(E_a) < \infty$.

Define $h = a(\chi_{E_a})$.

h is a bounded measurable function with finite support and $0 \leq h \leq f$ on E.

By definition of $\int_E f$: $\quad \int_E f \geq \int_E h = a\big(m(E_a)\big)$.

Thus $\quad m(\{x \in E \mid f(x) \geq a\}) \leq \frac{1}{a}\int_E f$.

Prop: Let f be a nonnegative measurable function on E. Then

$$\int_E f = 0 \text{ if and only if } f = 0 \text{ a.e. on } E.$$

Proof: Assume $\int_E f = 0$.

Let $E_{\frac{1}{n}} = \{x \in E \mid f(x) \geq \frac{1}{n}\}$.

Then by Chebychev's inequality:

$$m\left(E_{\frac{1}{n}}\right) \leq n\int_E f = 0 \quad \text{for all } n.$$

$\{x \in E \mid f(x) > 0\} = \bigcup_{n=1}^{\infty} E_{\frac{1}{n}}$ and $E_{\frac{1}{n}} \subseteq E_{\frac{1}{n+1}}$. Thus

$$\lim_{n\to\infty} m\left(\bigcup_{n=1}^{\infty} E_{\frac{1}{n}}\right) = \lim_{n\to\infty} m\left(E_{\frac{1}{n}}\right) = 0.$$

Hence $\quad m(\{x \in E \mid f(x) > 0\}) = 0$

and $f = 0$ a.e. on E.

Now suppose $f = 0$ a.e. on E.

Let φ be a simple function and h be a bounded measurable function of finite support for which $0 \le \varphi \le h \le f$ on E.

Since φ is simple, $\int_E \varphi = 0$, for any $\varphi \le h$, thus $\int_E h = 0$.

Since this holds for all $h \le f$, $\int_E f = 0$.

Theorem: Let f and g be nonnegative measurable functions on E. Then

1. for any $\alpha, \beta > 0$ $\int_E (\alpha f + \beta g) = \alpha \int_E f + \beta \int_E g$
2. if $f \le g$ on E then

$$\int_E f \le \int_E g.$$

Proof: First let's show $\int_E \alpha f = \alpha \int_E f$.

Let h be any bounded, measurable function of finite support and $0 \le h \le f$.

$0 \le h \le f$ if an only if $0 \le \alpha h \le \alpha f$.

Notice that $\int_E \alpha h = \alpha \int_E h$.

$$\int_E \alpha f = \sup\left\{\int_E \alpha h \,\middle|\, \alpha h \; bounded, measuable, of \; finite \; support \; and \; 0 \le \alpha h \le \alpha f \; on \; E\right\}$$

$$= \alpha \sup\left\{\int_E h \,\middle|\, h \; bounded, measuable, of \; finite \; support \; and \; 0 \le h \le f \; on \; E\right\}$$

$$= \alpha \int_E f \,.$$

To prove linearity we only need to show: $\int_E (f+g) = \int_E f + \int_E g$.

Let F, G be bounded, measurable functions of finite support with

$0 \le F \le f$ and $0 \le G \le g$.

Then $0 \le F + G \le f + g$ and $F + G$ is bounded, measurable, and finite support.

$\int_E F + \int_E G = \int_E (F+G) \le \int_E (f+g)$ for all $0 \le F \le f$ and $0 \le G \le g$ that are bounded, measurable, and of finite support.

Thus $\int_E f + \int_E g \le \int_E (f+g)$.

By definition:

$$\int_E (f+g) = \sup\left\{\int_E l \,\middle|\, l \; bounded, measurable, of \; finite \; support \; and \; 0 \le l \le f + g \; on \; E\right\}.$$

Let's show that $\int_E \, l \leq \int_E \, f + \int_E \, g$.

Let $h = \min\{f, l\}$ and $k = l - h$ on E.

Notice if $x \in E$ and $l(x) \leq f(x)$ then $k(x) = 0 \leq g(x)$.

If $l(x) > f(x)$ then $k(x) = l(x) - f(x) \leq g(x)$ since $0 \leq l \leq f + g$.

Thus $h(x) \leq g(x)$ on E.

Both h and k are bounded, measurable function of finite support.

Thus we have: $0 \leq h \leq f$, $0 \leq k \leq g$ and $l = h + k$ on E.

So $\int_E \, l = \int_E \, h + \int_E \, k \leq \int_E \, f + \int_E \, g$.

Thus $\int_E \, (f + g) \leq \int_E \, f + \int_E \, g$

and hence $\int_E \, (f + g) = \int_E \, f + \int_E \, g$.

To prove monotonicity: let h be any bounded, measurable function of finite support where $0 \leq h \leq f$ on E.

But since $f \leq g$ then $h \leq g$ on E.

By definition of $\int_E \, g$, $\int_E \, h \leq \int_E \, g$.

But $\int_E \, f = \sup\{\int_E \, h\}$ so $\int_E \, f \leq \int_E \, g$.

Theorem: Let f be a nonnegative measurable function on E. If A and B are disjoint measurable subsets of E, then

$$\int_{A\cup B} f = \int_A f + \int_B f.$$

In particular, if E_0 is a subset of E of measure 0, then

$$\int_E f = \int_{E\sim E_0} f.$$

Proof: The first relationship follows from: $f = (f)(\chi_A) + (f)(\chi_B)$ and the fact that $\int_E (f)(\chi_A) = \int_A f$.

$\int_E f = \int_{E\sim E_0} f$ follows from the first relationship and the fact that $\int_{E_0} f = 0$ since $m(E_0) = 0$.

Recall that if $\{a_n\}$ is a sequence of real numbers then the limit superior of $\{a_n\}$, denoted by limsup$\{a_n\}$, is given by:

$$\textbf{limsup}\{\boldsymbol{a_n}\} = \lim_{n\to\infty} \sup\{\boldsymbol{a_k} |\ \boldsymbol{k} \geq \boldsymbol{n}\}.$$

The limit inferior of $\{a_n\}$, denoted by liminf$\{a_n\}$, is given by:

$$\textbf{liminf}\{\boldsymbol{a_n}\} = \lim_{n\to\infty} \inf\{\boldsymbol{a_k} |\ \boldsymbol{k} \geq \boldsymbol{n}\}.$$

Another way to think of these notions is to take all subsequential limits of $\{a_n\}$ and call that set E.

$$\text{limsup}\{a_n\} = \sup(E)$$

$$\text{liminf}\{a_n\} = \inf(E)$$

If $\{a_n\}$ has a limit l, then limsup$\{a_n\} = \text{liminf}\{a_n\} = l$.

Ex. Let $a_{3n-2} = \frac{1}{n}$, $a_{3n-1} = 1 - \frac{1}{n}$, $a_{3n} = -\frac{n}{n+1}$.

$\{1,0,-\frac{1}{2},\frac{1}{2},\frac{1}{2},-\frac{2}{3},\frac{1}{3},\frac{2}{3},-\frac{3}{4},\ldots\}$; $E = \{-1, 0, 1\}$.

Thus $\text{limsup}\{a_n\} = \sup(E) = 1$ and $\text{liminf}\{a_n\} = \inf(E) = -1$.

Fatou's lemma: Let $\{f_n\}$ be a sequence of nonnegative measurable functions on E. If $f_n \to f$ pointwise a.e. on E, then

$$\int_E f \leq liminf \int_E f_n.$$

Proof: Since $\int_A f = 0$, if $m(A) = 0$, we can assume that $f_n \to f$ pointwise on all of E.

f is nonnegative and measurable because it's the pointwise limit of nonnegative, measurable functions.

To prove the theorem it is enough to prove that

$$\int_E h \leq liminf \int_E f_n$$

for any bounded, measurable function of finite support for which $0 \leq h \leq f$ on E.

Let h be a bounded, measurable function of finite support for which $0 \leq h \leq f$ and $|h| \leq M$ for some $M \geq 0$ on E.

Let $E_0 = \{x \in E |\, h(x) \neq 0\}$.

$m(E_0) < \infty$ since h has finite support.

Let $h_n = \min\{h, f_n\}$ on E.

Notice that h_n is measurable and $0 \le h_n \le M$, and $h_n \equiv 0$ on $E \sim E_0$.

If $x \in E$, since $h(x) \le f(x)$ and $f_n(x) \to f(x)$, $h_n(x) \to h(x)$.

Thus $h_n(x)$ is uniformly bounded (by M), and if we restrict $\{h_n\}$ to E_0, $m(E_0) < \infty$, then we can apply the bounded convergence theorem.

$$\lim_{n\to\infty} \int_E h_n = \lim_{n\to\infty} \int_{E_0} h_n = \int_{E_0} h = \int_E h$$

Since $h_n \equiv 0$ on $E \sim E_0$.

However, for each n, $h_n \le f_n$ on E.

Thus we have:

$$\int_E h_n \le \int_E f_n .$$

Hence:

$$\int_E h = \lim_{n\to\infty} \int_E h_n \le liminf \int_E f_n .$$

Since this is true for all h, nonnegative, measurable, bounded and $0 \le h \le f$ on E we have:

$$\int_E f \le liminf \int_E f_n.$$

Ex. Here's an example where you have a strict inequality in Fatou's lemma.

Let $E = [0,1]$ and let $f_n = (n)\chi_{(0,\frac{1}{n})}$.

$\{f_n\}$ converges to $f = 0$ on $[0,1]$.

However $\int_0^1 f_n = 1$ for all n, but $\int_0^1 f = 0$.

Thus $0 = \int_0^1 f < liminf \int_0^1 f_n = 1$.

If we add the condition that $\{f_n\}$ is monotonically increasing then the inequality in Fatou's lemma becomes an equality.

The Monotone Convergence Theorem: Let $\{f_n\}$ be an increasing sequence of nonnegative measurable functions on E. If $f_n \to f$ pointwise a.e. on E then

$$\lim_{n\to\infty} \int_E f_n = \int_E f.$$

Proof: According to Fatou's lemma: $\int_E f \leq liminf \int_E f_n$.

However, for each n, $f_n \leq f$ a.e. on E.

Thus $\int_E f_n \leq \int_E f$ for each n.

Therefore, $limsup \int_E f_n \leq \int_E f$.

Since limsup$\{a_n\} \geq liminf\{a_n\}$ we have:

$$liminf \int_E f_n \leq limsup \int_E f_n \leq \int_E f \leq liminf \int_E f_n.$$

Thus $\lim_{n\to\infty} \int_E f_n = \int_E f$.

Corollary: Let $\{u_n\}$ be a sequence of nonnegative measurable functions on E. If $f = \sum_{n=1}^{\infty} u_n$ pointwise a.e. on E, then $\int_E f = \sum_{n=1}^{\infty} \int_E u_n$.

Proof: Let $f_n = \sum_{k=1}^{n} u_k$.

Then $\{f_n\}$ is increasing, nonnegative and measurable.

Thus by the Monotone Convergence Theorem:

$$\lim_{n\to\infty} \int_E f_n = \int_E f$$

$$\lim_{n\to\infty} \int_E \sum_{k=1}^{n} u_k = \int_E f$$

$$\lim_{n\to\infty} \sum_{k=1}^{n} \int_E u_k = \sum_{k=1}^{\infty} \int_E u_k = \int_E f.$$

Ex. Let $f(x) = \frac{1}{1-x^4} = 1 + x^4 + x^8 + \cdots = \sum_{k=0}^{\infty} x^{4k}$. Evaluate $\int_0^{\frac{1}{2}} f$.

$$\int_0^{\frac{1}{2}} f = \int_0^{\frac{1}{2}} \sum_{k=0}^{\infty} x^{4k} = \sum_{k=0}^{\infty} \int_0^{\frac{1}{2}} x^{4k} \quad \text{(by the previous corollary)}$$

$$= \sum_{k=0}^{\infty} \frac{x^{4k+1}}{4k+1} \Big|_{x=0}^{x=\frac{1}{2}}$$

$$= \sum_{k=0}^{\infty} \frac{1}{(4k+1)(2^{(4k+1)})} .$$

Another application of the monotone convergence theorem is that it allows us to evaluate some Lebesgue integrals.

Ex. Evaluate the Lebesgue integral $\int_E \frac{1}{x^2}$ where $E = [1, \infty)$.

We know that if f is bounded and measurable on a closed bounded interval D, then if the Riemann integral exists over D, then it's equal to the Lebesgue integral over D.

Let $E_n = [1, n]$, $n \in \mathbb{Z}^+$, and $f_n = \frac{1}{x^2}$ if $x \in E_n$

$= 0$ if $x \in (n, \infty)$.

Then $\{f_n\}$ is bounded, increasing, nonnegative and measurable.

In addition, $f_n \to f$ pointwise on E.

Thus by the monotone convergence theorem: $\lim_{n\to\infty} \int_E f_n = \int_E f$.

But $\int_E f_n = \int_{E_n} f_n = \int_1^n \frac{1}{x^2}$ (where this is a Riemann integral).

$\int_1^n \frac{1}{x^2} = -\frac{1}{x} \Big|_{x=1}^{x=n} = 1 - \frac{1}{n}$

So $\lim_{n\to\infty} \int_E f_n = \lim_{n\to\infty} \left(1 - \frac{1}{n}\right) = 1 = \int_E f = \int_E \frac{1}{x^2}$.

Def. A nonnegative measurable function f on a measurable set E is said to be **integrable over E** if $\int_E f < \infty$.

Prop. Let the nonnegative function f be integrable over E. Then f is finite a.e. on E.

Proof: By Chebychev's inequality we know:

$$m(\{x \in E \mid f(x) \geq n\}) \leq \frac{1}{n}\int_E f.$$

By monotonicity we know:

$$m(\{x \in E \mid f(x) = \infty\}) \leq m(\{x \in E \mid f(x) > n\}) \leq \frac{1}{n}\int_E f.$$

But $\int_E f < \infty,$ so $m(\{x \in E \mid f(x) = \infty\}) = 0,$thus f is finite a.e. on E.

Beppo Levi's Lemma: Let $\{f_n\}$ be an increasing sequence of nonnegative measurable functions on E. If the sequence of integrals $\{\int_E f_n\}$ is bounded, then $\{f_n\}$ converges pointwise on E to a measurable function f that is finite a.e. on E and $\lim_{n\to\infty} \int_E f_n = \int_E f < \infty$.

Proof: Let $f(x) = \lim_{n\to\infty} f_n(x)$ for $x \in E$.

By monotone convergence: $\lim_{n\to\infty} \int_E f_n = \int_E f < \infty$;

since $\{\int_E f_n\}$ is bounded.

f is finite a.e. on E since $\int_E f < \infty$.