Lebesgue Outer Measure

The **Lebesgue measure** of a subset of $\mathbb{R}$ is a generalization of the length of a set. We want a Lebesgue measure, m, to satisfy the following three properties:

1) Each nonempty interval $I \subseteq \mathbb{R}$ is Lebesgue measurable and $m(I) = l(I) =$ length of I.

2) m is translation invariant. That is, If E is a Lebesgue measureable set and $t \in \mathbb{R}$, then the translate of E by t, $E + t = \{x + t |\ x \in E\}$, is also Lebesgue measurable and $m(E + t) = m(E)$.

3) If $\{E_k\}, k = 1, 2, \ldots, \infty$ is a countable disjoint collection of Lebesgue measurable sets then

$$m\left(\bigcup_{k=1}^{\infty} E_k\right) = \sum_{k=1}^{\infty} m(E_k).$$

Unfortunately, it's not possible to create a set function that possesses all three properties and is defined for all subsets of $\mathbb{R}$. In fact, there is not even a set function defined for all subsets of $\mathbb{R}$ that satisfies 1 and 2 and is finitely additive.

To construct the Lebesgue measure we will start by defining a set function called an **outer measure**, denoted by m^*, that is defined on all subsets of $\mathbb{R}$, satisfies properties 1 and 2, but is countably subadditive, that is, for any collections of subsets of $\mathbb{R}$, E_i, disjoint or not

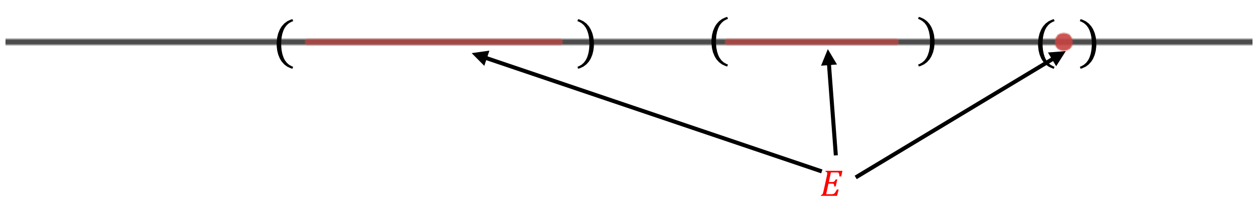
$$m^*\left(\bigcup_{i=1}^{\infty} E_i\right) \leq \sum_{k=1}^{\infty} m^*(E_i).$$

We will then determine what it means for a set to be Lebesgue measurable and show that the collection of Lebesgue measurable sets forms a σ-algebra (i.e. it contains $\mathbb{R}$ and is closed with respect to complements and countable unions) containing the open and closed sets. We will then restrict m^* to this collection of sets and denote it by m and prove m is countably additive. m will be the Lebesgue measure.

We start by defining the length of an interval (closed, open, or half closed/open) I, $l(I)$, to be $|b - a|$, where a, b are the endpoints, if both a and b are finite and ∞ if either a or b is not finite.

If A is a set of real numbers, consider $\{I_k\}, k = 1, 2, \ldots, \infty$, where I_k is an open, bounded interval and $A \subseteq \bigcup_{k=1}^{\infty} I_k$. We define the outer measure of E, $m^*(E)$ to be:

$$m^*(E) = \inf\{\sum_{k=1}^{\infty} l(I_k) \mid E \subseteq \bigcup_{k=1}^{\infty} I_k\}$$



Notice:

a) $m^*(\phi) = 0$.

b) If $E \subseteq F$, then $m^*(E) \leq m^*(F)$ because any cover of F is also a cover of E.

Ex. Any countable set A has $m^*(A) = 0$

Let $A = \{a_1, a_2, a_3, \ldots\}$ and let $I_k = (a_k - \frac{\epsilon}{2^{k+1}},\ a_k + \frac{\epsilon}{2^{k+1}})$.

Then $0 \le m^*(A) \le \sum_{k=1}^{\infty} l(I_k) = \sum_{k=1}^{\infty} \frac{\epsilon}{2^k} = \epsilon$.

This holds for all $\epsilon > 0$ hence $m^*(A) = 0$.

Ex. $m^*(\mathbb{Q}) = 0,\ \ m^*(\mathbb{Z}) = 0$.

Prop: $m^*(I) = l(I)$

Proof: First let's show this for a bounded interval $[a, b]$.

Since the open interval $(a - \epsilon,\ b + \epsilon)$ contains $[a, b]$ for all $\epsilon > 0$ we have

$$m^*([a, b]) \le b - a + 2\epsilon.$$

Since this is true for all $\epsilon > 0$

$$m^*([a, b]) \le b - a.$$

Now let's show $m^*([a, b]) \ge b - a$:

Let $\{I_k\}$ be a set of open, bounded intervals such that:

$$\bigcup_{k=1}^{\infty} I_k \supseteq [a, b].$$

We will show that:

$$\sum_{k=1}^{\infty} l(I_k) \ge b - a.$$

By the Heine-Borel Theorem any covering of $[a, b]$ by open intervals has a finite subcover, $\{I_k\}, k = 1, \dots, n$. Now let's show:

$$\sum_{k=1}^{n} l(I_k) \geq b - a.$$

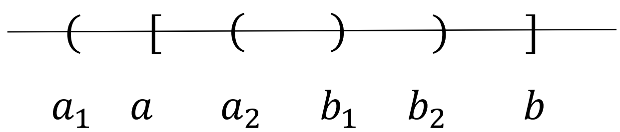
Since $a \in \bigcup_{k=1}^{\infty} I_k$, there is at least one I_k with $a \in I_k$.

Let's call this I_k, (a_1, b_1) where $a_1 < a < b_1$.

If $b_1 \geq b$ then $l(I_k) \geq b_1 - a_1 > b - a$ and:

$$\sum_{k=1}^{n} l(I_k) \geq b_1 - a_1 > b - a.$$

Otherwise $b_1 \in [a, b)$, and since $b_1 \notin (a_1, b_1)$ there exists an



interval in the collection $\{I_k\}, k = 1, 2, \dots, n$, call it (a_2, b_2) distinct from (a_1, b_1) for which $b_1 \in (a_2, b_2)$, that is $a_2 < b_1 < b_2$.

If $b_2 \geq b$ then:

$$\sum_{k=1}^{n} l(I_k) \geq (b_1 - a_1) + (b_2 - a_2)$$
$$= b_2 - (a_2 - b_1) - a_1 > b_2 - a_1 > b - a.$$

Continue this process until it terminates (it must because n, the number of open intervals is finite).

Thus there is a subcollection $\{(a_k, b_k)\}, k = 1, \dots, m$ of $\{I_k\}, k = 1, \dots n$ for which $a_1 < a$ while $a_{k+1} < b_k$ for $1 \leq k \leq m - 1$ and $b_m > b$.

Thus $\sum_{k=1}^{n} l(I_k) \geq \sum_{k=1}^{m} l(a_k, b_k)$

$$= (b_m - a_m) + (b_{m-1} - a_{m-1}) + \cdots + (b_1 - a_1)$$

$$= b_m - (a_m - b_{m-1}) - \cdots - (a_2 - b_1) - a_1$$

$$> b_m - a_1 > b - a.$$

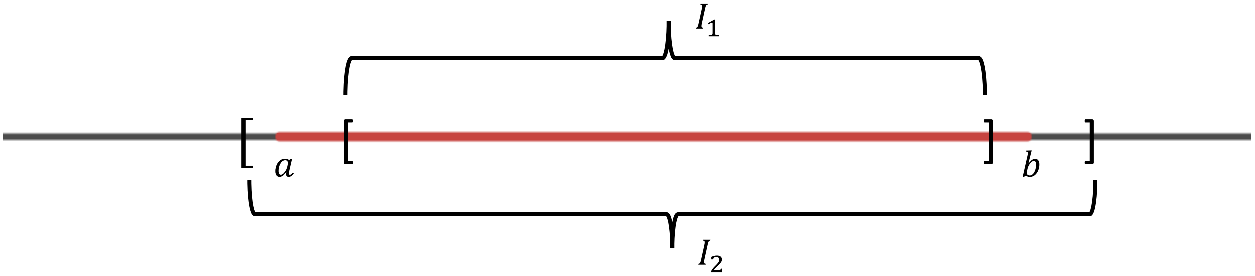
Hence $\sum_{k=1}^{n} l(I_k) \geq b - a$ and so $\sum_{k=1}^{\infty} l(I_k) \geq b - a$.

Thus $m^*([a, b]) = b - a$.

If I is any bounded interval $((a, b), [a, b), (a, b])$, then given any $\epsilon > 0$ there are two closed, bounded intervals I_1, I_2 such that

$$I_1 \subseteq I \subseteq I_2$$

while, $l(I) - \epsilon < l(I_1)$ and $l(I_2) < l(I) + \epsilon$.



Thus $l(I) - \epsilon < l(I_1) = m^*(I_1) \leq m^*(I) \leq m^*(I_2) = l(I) + \epsilon$

since if $A \subseteq B$ then $m^*(A) \leq m^*(B)$.

This holds for all $\epsilon > 0$, thus $l(I) = m^*(I)$.

If I is unbounded, then for each natural number n, there is an interval $J \subseteq I$ with $l(J) = n$. Hence $m^*(I) \geq m^*(J) = l(J) = n$.

Thus $m^*(I) = \infty$.

Prop : $m^*(A + t) = m^*(A)$, for any $t \in \mathbb{R}$.

Proof: If $\{I_k\}, k = 1, 2, \ldots, \infty$ is any collection of intervals, then $\{I_k\}$ covers A if, and only if, $\{I_k + t\}, k = 1, 2, \ldots, \infty$ covers $A + t$.

Notice also if I_k is an open interval, so is $I_k + t$, and it has the same length.

Thus, $\sum_{k=1}^{\infty} l(I_k) = \sum_{k=1}^{\infty} l(I_k + t)$.

So $m^*(A + t) = m^*(A)$.

Prop: m^* is countably subadditive, i.e., if $\{E_k\}, k = 1, 2, \ldots, \infty$ is any countable collection of sets, disjoint or not, then:

$$m^*(\bigcup_{k=1}^{\infty} E_k) \leq \sum_{k=1}^{\infty} m^*(E_k).$$

Proof: If one of the E_k's has $m^*(E_k) = \infty$, then the inequality is obviously true.

So assume $m^*(E_k)$ is finite for all k.

Let $\epsilon > 0$.

For each k, there is a countable collection $\{I_{k,j}\}, j = 1, \dots, \infty$ of open bounded intervals for which:

$$E_k \subseteq \bigcup_{j=1}^{\infty} I_{k,j} \text{ and } \sum_{j=1}^{\infty} l(I_{k,j}) < m^*(E_k) + \frac{\epsilon}{2^k}.$$

$\{I_{k,j}\}, j, k = 1, \dots, \infty$ is a countable collection of open bounded intervals that covers $\bigcup_{k=1}^{\infty} E_k$.

Thus:

$$\begin{aligned} m^*(\cup_{k=1}^{\infty} E_k) &\leq \sum_{1 \leq k,j < \infty} l(I_{k,j}) \\ &= \sum_{k=1}^{\infty} \sum_{j=1}^{\infty} l(I_{k,j}) \\ &\leq \sum_{k=1}^{\infty} (m^*(E_k) + \frac{\epsilon}{2^k}) \\ &= \sum_{k=1}^{\infty} m^*(E_k) + \epsilon. \end{aligned}$$

Since this holds for all $\epsilon > 0$,

$$m^*(\cup_{k=1}^{\infty} E_k) \leq \sum_{k=1}^{\infty} m^*(E_k).$$

Clearly, finite subadditivity follows from countable subadditivity (just let $E_k = \phi$ for $k > n$).