Isomorphisms- HW Problems

For problems 1-7 determine if the linear transformation is an isomorphism. Explain your answer.

1. $T: \mathbb{R}^2 \to \mathbb{R}^3$ by $T(x_1, x_2) = (x_1,\ x_2,\ x_1 + x_2)$
2. $T: \mathbb{R}^3 \to \mathbb{R}^2$ by $T(x_1, x_2, x_3) = (x_1 + 2x_2,\ 2x_1 - x_2)$
3. $T: \mathbb{R}^3 \to \mathbb{R}^3$ by $T(x_1, x_2, x_3) = (x_1 + x_3,\ x_1 + x_2,\ x_2 + x_3)$
4. $T: \mathbb{R}^3 \to \mathbb{R}^3$ by $T(x_1, x_2, x_3) = (x_3 - x_2,\ x_1 + 2x_2,\ 3x_3 - x_2 - x_1)$
5. $T: P_2(\mathbb{R}) \to \mathbb{R}^3$ by $T(a_0 + a_1x + a_2x^2) = (a_0,\ a_1,\ a_2)$
6. $T: M_{2\times2}(\mathbb{R}) \to P_3(\mathbb{R})$ by

$$T\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = a + (b + c)x + (c + d)x^2 + dx^3$$

7. $T: M_{2\times2}(\mathbb{R}) \to M_{2\times2}(\mathbb{R})$ by $T\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = \begin{bmatrix} d & -c \\ -b & a \end{bmatrix}$.

8. Suppose $A, B \in M_{n\times n}(\mathbb{R})$ where A, B are both invertible. Prove that AB is invertible and $(AB)^{-1} = B^{-1}A^{-1}$.

9. Let $A, B \in M_{n\times n}(\mathbb{R})$ where A is invertible. Suppose $AB = 0$ (the zero matrix). Prove that $B = 0$.

10. Suppose $T, S: V \to W$ are linear transformations of the vector space V onto W. Suppose there is a vector $v \in V,\ v \neq 0$ such that $T(v) = S(v)$. Prove that the linear transformation $U = T - S$ is not an isomorphism.