Composition of Linear Transformations

Theorem: Let $V, W,$ and Z be vector spaces and let $T: V \to W$ and $U: W \to Z$ be linear then $U \circ T = UT: V \to Z$ is linear.

Proof: Let $u, v \in V$ and $c \in \mathbb{R}$ then

$$\begin{aligned} UT(cu+v) &= U(T(cu+v)) \\ &= U(cT(u)+T(v)) \\ &= U\big(cT(u)\big)+U(T(v)) \\ &= cU\big(T(u)\big)+U(T(v)) \\ &= cUT(u)+UT(v). \end{aligned}$$

Thus UT is linear.

Theorem: Let V be a vector space. Let $T, U_1, U_2 \in \mathcal{L}(V)$ then

a. $T(U_1+U_2) = TU_1 + TU_2$ and $(U_1+U_2)T = U_1T + U_2T$

b. $T(U_1U_2) = (TU_1)U_2$

c. $TI = IT = T$.

d. $c(U_1U_2) = (cU_1)U_2 = U_1(cU_2)$ for any $c \in \mathbb{R}$.

Proof of a.: Let $v \in V$ then

$$\begin{aligned} (T(U_1+U_2))(v) &= T(U_1(v)+U_2(v)) \\ &= TU_1(v)+TU_2(v) \\ &= (TU_1+TU_2)(v). \end{aligned}$$

Similarly for $(U_1+U_2)T = U_1T + U_2T$.

Let $T: V \to W$ and $U: W \to Z$ be linear transformations of vector spaces. The key fact is that if we have matrix representations of T and U as A and C respectively, then the matrix representation of the composition UT is CA. That is the matrix product of C and A.

We can see this by letting $B_1 = \{v_1, \dots, v_n\}$, $B_2 = \{w_1, \dots w_m\}$, and $B_3 = \{z_1, \dots, z_p\}$ be ordered bases for V, W, and Z respectively. Thus for any basis vector $v_j \in V$ we have:

$$
\begin{aligned}
UT(v_j) = U\left(T(v_j)\right) &= U(\textstyle\sum_{k=1}^{m} A_{kj} w_k) \\
&= \textstyle\sum_{k=1}^{m} A_{kj} U(w_k) \\
&= \textstyle\sum_{k=1}^{m} A_{kj} (\sum_{i=1}^{p} C_{ik} z_i) \\
&= \textstyle\sum_{i=1}^{p} (\sum_{k=1}^{m} C_{ik} A_{kj}) z_i \\
&= \textstyle\sum_{i=1}^{p} D_{ij} z_i
\end{aligned}
$$

where $D_{ij} = \sum_{k=1}^{m} C_{ik} A_{kj}$.

Def. Let C be an $m \times n$ matrix and A an $n \times p$ matrix. We define the **product of $\boldsymbol{C}$ and $\boldsymbol{A}$** to be

$$(CA)_{ij} = \sum_{k=1}^{m} C_{ik} A_{kj} \quad \text{for } 1 \le i \le m,\ 1 \le j \le p.$$

Thus if the matrix representation of $T: V \to W$ is A and the matrix representation of $U: W \to Z$ is C then the matrix representation of $UT: V \to Z$ is CA.

Ex. Let $T: \mathbb{R}^2 \to \mathbb{R}^3$ and $U: \mathbb{R}^3 \to \mathbb{R}^2$ be linear transformations defined by

$$T(< a_1, a_2 >) =< 2a_1 - a_2,\ a_2,\ a_2 - 3a_1 >$$

$$U(< b_1, b_2, b_3 >) =< b_1 + b_2 + 2b_3,\ b_2 - 3b_3 >$$

with respect to the standard ordered bases B_1 for $\mathbb{R}^2$ and B_2 for $\mathbb{R}^3$. Find a matrix representation of UT with respect to these bases.

We saw in an earlier example that

$$[T]_{B_1}^{B_2} = \begin{bmatrix} 2 & -1 \\ 0 & 1 \\ -3 & 1 \end{bmatrix}.$$

To find the matrix representation of U notice that

$U(< 1,0,0 >) =< 1,0 >$

$U(< 0,1,0 >) =< 1,1 >$

$U(< 0,0,1 >) =< 2,-3 >$,

Thus $\qquad [U]_{B_2}^{B_1} = \begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & -3 \end{bmatrix}.$

So we have: $[UT]_{B_1}^{B_1} = \begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & -3 \end{bmatrix} \begin{bmatrix} 2 & -1 \\ 0 & 1 \\ -3 & 1 \end{bmatrix}$

$$= \begin{bmatrix} -4 & 2 \\ 9 & -2 \end{bmatrix}.$$

Notice that we can also calculate this by directly finding a matrix representation of $U \circ T$ from the formulas for $T(< a_1, a_2 >)$ and $U(< b_1, b_2, b_3 >)$.

$$\begin{aligned} U \circ T(< a_1, a_2 >) &= U(T(< a_1, a_2 >) \\ &= U(< 2a_1 - a_2,\ \ a_2,\ \ a_2 - 3a_1 >) \\ &=< 2a_1 - a_2 + a_2 + 2(a_2 - 3a_1),\ a_2 - 3(a_2 - 3a_1) > \\ &=< -4a_1 + 2a_2,\ 9a_1 - 2a_2 >. \end{aligned}$$

Thus we have:

$U \circ T(< 1,0 >) =< -4, 9 >$ $\qquad [U \circ T\,]_{B_1}^{B_1} = \begin{bmatrix} -4 & 2 \\ 9 & -2 \end{bmatrix}.$

$U \circ T(< 0,1 >) =< 2,\ -2 >$

Ex. Let $U: P_3(\mathbb{R}) \to P_2(\mathbb{R})$ and $T: P_2(\mathbb{R}) \to P_3(\mathbb{R})$ be linear transformations defined by $U\big(p(x)\big) = p'(x)$ and $T\big(q(x)\big) = \int_0^x q(t)dt$. Let B_1 and B_2 be the standard bases for $P_3(\mathbb{R})$ and $P_2(\mathbb{R})$ respectively. Find the matrix representation of $UT: P_2(\mathbb{R}) \to P_2(\mathbb{R})$ and $TU: P_3(\mathbb{R}) \to P_3(\mathbb{R})$.

Let's first find the matrix representations of T and U.

$B_1 = \{1, x, x^2, x^3\}, \quad B_2 = \{1, x, x^2\}$.

$T: P_2(\mathbb{R}) \to P_3(\mathbb{R})$ by $T\big(q(x)\big) = \int_0^x q(t)dt$,

$T(1) = \int_0^x 1dt = x = \quad 0(1) + 1(x) + 0(x^2) + 0(x^3)$

$T(x) = \int_0^x tdt = \frac{1}{2}(x^2) = \quad 0(1) + 0(x) + \frac{1}{2}(x^2) + 0(x^3)$

$T(x^2) = \int_0^x t^2dt = \left(\frac{1}{3}\right)x^3 = 0(1) + 0(x) + 0(x^2) + \frac{1}{3}(x^3)$

$$\Longrightarrow \qquad [T]_{B_2}^{B_1} = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & 0 & \frac{1}{3} \end{bmatrix}.$$

$U: P_3(\mathbb{R}) \to P_2(\mathbb{R})$ by $U\big(p(x)\big) = p'(x), \quad B_1 = \{1, x, x^2, x^3\}$.

$U(1) = 0 = \quad 0(1) + 0(x) + 0(x^2)$

$U(x) = 1 = \quad 1(1) + 0(x) + 0(x^2)$

$U(x^2) = 2x = 0(1) + 2(x) + 0(x^2)$

$U(x^3) = 3x^2 = 0(1) + 0(x) + 3(x^2)$

$$\Longrightarrow \qquad [U]_{B_1}^{B_2} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \end{bmatrix}.$$

$$[UT]_{B_2}^{B_2} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & 0 & \frac{1}{3} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

$$[TU]_{B_1}^{B_1} = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & 0 & \frac{1}{3} \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

Def. The **Kronecker delta**, δ_{ij}, is defined by

$$\delta_{ij} = 1 \quad \text{if } i = j$$
$$= 0 \quad \text{if } i \neq j.$$

The $n \times n$ **identity matrix** I_n is defined by

$$(I_n)_{ij} = \delta_{ij}.$$

For example:

$$I_1 = [1], \qquad I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \qquad I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \text{ etc.}$$

Theorem: Let A be an $m \times n$ matrix, B and C be $n \times p$ matrices and D and E $q \times m$ matrices. Then

a. $A(B + C) = AB + AC$ and $(D + E)A = DA + EA$.

b. $c(AB) = (cA)B = A(cB)$, $c \in \mathbb{R}$.

c. $I_m(A) = A = A(I_n)$

d. if V is an n-dimensional vector space with ordered basis β, then $[I_V]_\beta = I_n$.

All of these results follow from straight forward matrix calculations.

Corollary: Let A be an $m \times n$ matrix, $B_1, \dots, B_k$ $n \times p$ matrices, $C_1, \dots, C_k$ $q \times m$ matrices and $a_1, \dots, a_k \in \mathbb{R}$, then

$$A\left(\sum_{i=1}^{k} a_i B_i\right) = \sum_{i=1}^{k} a_i A B_i \qquad \text{and}$$

$$\left(\sum_{i=1}^{k} a_i C_i\right) A = \sum_{i=1}^{k} a_i C_i A\,.$$

Def. For an $n \times n$ matrix A we define

$$A^1 = A$$

$$A^2 = A(A)$$

$$A^3 = A^2(A)$$

$$\vdots$$

$$A^k = A^{k-1}(A).$$

Notice that unlike real numbers, if A is a matrix and $A^2 = 0$ (ie the zero matrix)

It does not imply that $A = 0$. For example if $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ then

$$A^2 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

Theorem: Let V and W be finite dimensional vector spaces with ordered bases B_1 and B_2 respectively and let $T: V \to W$ be linear. Then for $u \in V$, we have

$$[T(u)]_{B_2} = [T]_{B_1}^{B_2}[u]_{B_1}.$$

Proof: Fix $u \in V$ and let $U: \mathbb{R} \to V$ by $U(c) = cu$.

Notice that $U(1) = u$.

So the matrix representation of U is $\begin{bmatrix} \lambda_1 \\ \vdots \\ \lambda_n \end{bmatrix}$ where $u = \lambda_1 v_1 + \cdots + \lambda_n v_n$.

Now $TU(1) = T(u)$ and $[T(u)]_{B_2} = [T]_{B_1}^{B_2}[u]_{B_1}$.

In other words, if we have a matrix representation of a linear transformation T of V into W, then to calculate $T(u)$ for $u \in V$, we multiply the matrix representation of $T$with the column vector $u = \lambda_1 v_1 + \cdots + \lambda_n v_n$.

Ex. Let $T: \mathbb{R}^2 \to \mathbb{R}^3$ be given $T(< a_1, a_2 >) =< 2a_1 - a_2,\ a_2,\ a_2 - 3a_1 >$ with respect to the standard ordered bases B_1 for $\mathbb{R}^2$ and B_2 for $\mathbb{R}^3$. Find $T(u)$ where $u =< 5,2 >$ by matrix multiplication.

We saw in an earlier example that the matrix representation of T is given by

$$[T]_{B_1}^{B_2} = \begin{bmatrix} 2 & -1 \\ 0 & 1 \\ -3 & 1 \end{bmatrix}.$$

$[u]_{B_1} = \begin{bmatrix} 5 \\ 2 \end{bmatrix}$, so we have:

$$[T(u)]_{B_2} = [T]_{B_1}^{B_2}[u]_{B_1} = \begin{bmatrix} 2 & -1 \\ 0 & 1 \\ -3 & 1 \end{bmatrix}\begin{bmatrix} 5 \\ 2 \end{bmatrix} = \begin{bmatrix} 8 \\ 2 \\ -13 \end{bmatrix}.$$

Ex. Let $T: P_3(\mathbb{R}) \to P_2(\mathbb{R})$ by $T\big(p(x)\big) = p'(x)$. Find $T(3 + 3x - x^2 + 2x^3)$ through matrix multiplication (assume the standard ordered bases).

From an earlier example we know that

$$[T]_{B_1}^{B_2} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \end{bmatrix}.$$

$$[3 + 3x - x^2 + 2x^3]_{B_1} = \begin{bmatrix} 3 \\ 3 \\ -1 \\ 2 \end{bmatrix},$$ so we have:

$$[T(3 + 3x - x^2 + 2x^3)]_{B_2} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} 3 \\ 3 \\ -1 \\ 2 \end{bmatrix} = \begin{bmatrix} 3 \\ -2 \\ 6 \end{bmatrix}.$$

So $T(3 + 3x - x^2 + 2x^3) = 3(1) - 2(x) + 6(x^2) = 3 - 2x + 6x^2$.