Linear Transformations

Def. Let V and W be vector spaces. We call a function $T: V \to W$ a **linear transformation** from V to W if for all $u, v \in V$ and $c \in \mathbb{R}$

a. $T(u + v) = T(u) + T(v)$

b. $T(cv) = cT(v)$.

Theorem: Let $T: V \to W$ be a linear transformation from a vector space V to a vector space W. Then for $u, v, u_1, \dots u_n \in V$, $a, b, a_1, \dots, a_n \in \mathbb{R}$

1. $T(0) = 0$
2. $T(v - u) = T(v) - T(u)$
3. $T(au + bv) = aT(u) + bT(v)$
4. $T(\sum_{i=1}^{n} a_i u_i) = \sum_{i=1}^{n} a_i T(u_i)$

Proof:

1. $T(0) = T\big(2(0)\big) = 2T(0) \implies T(0) = 0.$

2. $T(v - u) = T\big(v + (-u)\big) = T(v) + T(-u)$

$$= T(v) - T(u).$$

3. $T(au + bv) = T(au) + T(bv) = aT(u) + bT(v).$

4. $T(\sum_{i=1}^{n} a_i u_i) = T(a_1 u_1 + a_2 u_2 + \cdots a_n v_n)$

$$= T(a_1 u_1) + T(a_2 u_2 + \cdots + a_n u_n)$$

$$= a_1 T(u_1) + T(a_2 u_2) + T(a_3 u_3 + \cdots a_n u_n)$$

$$\vdots$$

$$= a_1 T(u_1) + a_2 T(u_2) + \cdots + a_n T(u_n)$$

$$= \sum_{i=1}^{n} a_i T(u_i).$$

Ex. Show that $T: V \to W$ is a linear transformation if and only if
$T(cv + u) = cT(v) + T(u)$ for all $u, v \in V, \ \ c \in \mathbb{R}$.

Proof: Case #3 of the previous theorem shows if T is linear then

$$T(cv + u) = cT(v) + T(u) \text{ for all } u, v \in V, \ \ c \in \mathbb{R}.$$

Now let's show that if $T(cv + u) = cT(v) + T(u)$ for all $u, v \in V, \ \ c \in \mathbb{R}$

then T is linear.

We must show:

1. $T(u + v) = T(u) + T(v)$ for all $u, v \in V$
2. $T(cv) = cT(v)$ for all $c \in \mathbb{R}$.

1. Since $T(cv + u) = cT(v) + T(u)$ for all $u, v \in V, \ \ c \in \mathbb{R}$, it's true for $c = 1$.

   Thus $T(v + u) = T(v) + T(u)$ for all $u, v \in V$.

2. If we take $u = 0$ then $T(cv + 0) = cT(v) + T(0)$

$$\Longrightarrow \qquad T(cv) = cT(v) + 0 = cT(v).$$

Ex. Show that $T: \mathbb{R}^2 \to \mathbb{R}^2$ by $T(< a_1, a_2 >) =< a_1 + 2a_2, a_1 >$ is a linear transformation.

By the previous example we just need to show that $T(cv + u) = cT(v) + T(u)$ for all $u, v \in \mathbb{R}^2,\ \ c \in \mathbb{R}$.

For any $u, v \in \mathbb{R}^2$, we have $u =< x_1, y_1 >$, $v =< x_2, y_2 >$ and

$$T(c < x_1, y_1 > +< x_2, y_2 >) = T(< cx_1 + x_2,\ cy_1 + y_2 >)$$
$$=< cx_1 + x_2 + 2(cy_1 \ + y_2),\ cx_1 + x_2 >.$$

$$cT(< x_1, y_1 >) + T(< x_2, y_2 >) = c(< x_1 + 2y_1,\ x_1 >) + (< x_2 + 2y_2,\ x_2 >)$$
$$=< cx_1 + 2cy_1 + x_2 + 2y_2,\ cx_1 + x_2 >$$
$$=< cx_1 + x_2 + 2(cy_1 + y_2),\ cx_1 + x_2 >$$
$$= T(c < x_1, y_1 > +< x_2, y_2 >).$$

So $T(cu + v) = cT(u) + T(v)$ for all $u, v \in \mathbb{R}^2,\ \ c \in \mathbb{R}$ and T is linear.

Ex. Define $T: \mathbb{R}^2 \to \mathbb{R}^2$ by $T(< a_1, a_2 >) =< -a_1, a_2 >$. T is a reflection about the y-axis. Show that T is a linear transformation.

For any $u, v \in \mathbb{R}^2$, we have $u =< x_1, y_1 >$, $v =< x_2, y_2 >$ and

$$T(c < x_1, y_1 > +< x_2, y_2 >) = T(< cx_1 + x_2,\ cy_1 + y_2 >)$$
$$=< -(cx_1 + x_2),\ cy_1 + y_2 >.$$

$$cT(< x_1, y_1 >) + T(< x_2, y_2 >) = c(< -x_1, y_{1>}) +< -x_2, y_2 >$$
$$=< -cx_1 - x_2,\ y_1 + y_2 >$$
$$=< -(cx_1 + x_2),\ cy_1 + y_2 >$$
$$= T(c < x_1., y_1 > +< x_2, y_2 >)$$

So T is linear.

Ex. Show that $T: \mathbb{R}^2 \to \mathbb{R}^2$ by $T(< a, b >) =< a + 3, b >$ is not a linear transformation.

Notice that if $c \neq 1$ then $T(cv) \neq cT(v)$:

$$T(c < a, b >) = T(< ca, cb >) =< ca + 3, cb >$$

$$cT(< a, b >) = c(a + 3, b) = (c(a + 3), cb) \neq (ca + 3, cb).$$

It's also true that $T(< 0,0 >) \neq< 0,0 >$ and $T(u + v) \neq T(u) + T(v)$ in general.

Ex. Show that $T: \mathbb{R}^2 \to \mathbb{R}^2$ by $T(< a, b >) =< a, 0 >$ (called a **projection**) is a linear transformation.

If we let $u =< a_1, b_1 >$, $v =< a_2, b_2 >$ and $c \in \mathbb{R}$ then we have:

$$T(cu + v) = T(c < a_1, b_1 > +< a_2, b_2 >)$$
$$= T(< ca_1 + a_2,\ b_1 + b_2 >)$$
$$=< ca_1 + a_2, 0 >$$

$$cT(u) + T(v) = cT(< a_1, b_1 >) + T(< a_2, b_2 >)$$
$$= c < a_1, 0 > +< a_2, 0 >$$
$$=< ca_1 + a_2, 0 >$$

Thus $T(cu + v) = cT(u) + T(v)$ for all $u, v \in \mathbb{R}^2$, $c \in \mathbb{R}$, so T is linear.

Ex. Show that $T: P_n(\mathbb{R}) \to P_n(\mathbb{R})$ by $T\big(f(x)\big) = f'(x)$ is a linear transformation.

Let $f(x), g(x) \in P_n(\mathbb{R})$ and $c \in \mathbb{R}$.

$$T\big(cf(x) + g(x)\big) = (cf(x) + g(x))'$$
$$= cf'(x) + g'(x)$$
$$= cT\big(f(x)\big) + T(g(x)).$$

So T is a linear transformation.

Ex. Let $V = C[a,b]$, the vector space of continuous real valued function on $[a,b]$. Define $T: V \to \mathbb{R}$ by $T\big(f(x)\big) = \int_a^b f(x)dx$. Show that T is a linear transformation.

Let $f(x), g(x) \in C[a,b]$ and $c \in \mathbb{R}$ then

$$T\big(cf(x) + g(x)\big) = \int_a^b \big(cf(x) + g(x)\big)dx$$

$$= c\int_a^b f(x)dx + \int_a^b g(x)dx$$

$$= cT\big(f(x)\big) + T(g(x)).$$

So T is a linear transformation.

Ex. Show that $T: \mathbb{R}^2 \to \mathbb{R}^2$ by $T(< a_1, a_2 >) =< a_1a_2, a_2 >$ is not a linear transformation.

Let $v =< a_1, a_2 >$ and $c \in \mathbb{R}$, $c \neq 1$ then

$$T(cv) = T(c < a_1, a_2 >)$$

$$= T(< ca_1,\ ca_2 >)$$

$$=< c^2a_1a_2,\ ca_2 >$$

$$cT(< a_1, a_2 >) = c < a_1a_2, a_2 >$$

$$=< ca_1a_2, ca_2 >\neq T(cv).$$

Thus T is not a linear transformation.

In this case it also happens to be true that $T(u + v) \neq T(u) + T(v)$ in general.

Ex: Suppose $T: \mathbb{R}^2 \to \mathbb{R}^3$ is a linear transformation such that
$T(<2,3>) = <0,3,4>$ and $T(<3,2>) = <-1,2,3>$. Find $T(<2,8>)$.

Notice that we can write $<2,8>$ as a linear combination of $<2,3>$ and $<3,2>$.

$$a<2,3> + b<3,2> = <2,8>$$

$$<2a+3b, 3a+2b> = <2,8>$$

$$\Longrightarrow \qquad 2a + 3b = 2$$

$$3a + 2b = 8.$$

Solving these simultaneous equations we get $a = 4, b = -2$.

Thus we have: $4<2,3> -2<3,2> = <2,8>$.

Hence we have:

$$T(<2,8>) = T(4<2,3> -2<3,2>)$$

$$= 4T(<2,3>) - 2T(<3,2>)$$

$$= 4<0,3,4> -2<-1,2,3>$$

$$= <0,12,16> - <-2,4,6>$$

$$= <2,8,10>.$$

In fact, given any $<x,y> \in \mathbb{R}^2$ we can find $T(<x,y>)$ by writing $<x,y>$ as a linear combination of $<2,3>$ and $<3,2>$. In this case we would need to solve:

$$a<2,3> + b<3,2> = <x,y>$$

$$\Longrightarrow \qquad 2a + 3b = x$$

$$3a + 2b = y$$

for a and b in terms of x and y.

Two important linear transformations are:

1. The identity linear trasformation $I: V \to V,$ where $I(v) = v$ for all $v \in V$.
2. The zero linear transformation $T_0: V \to V,$ where $T_0(v) = 0$ for all $v \in V$.

Def. Let V and W be vector spaces, and let $T: V \to W$ be linear. The **null space** or **kernel** of $T, N(T),$ is the set of $v \in V$ such that $T(v) = 0$. The **range** or **image** of T is the subset of W given by $R(T) = \{T(v) |\ v \in V\}$.

Ex. Let $I: V \to V$ and $T_0: V \to W$ be the identity and zero transformations. Then

$$N(I) = \{0\} \qquad N(T_0) = V$$

$$R(I) = V \qquad R(T_0) = \{0\}.$$

Ex. Let $T: \mathbb{R}^3 \to \mathbb{R}^2$ be the linear transformation given by

$T(< a_1, a_2, a_3 >) =< a_1 + a_2, 3a_3 >$. Find $N(T)$ and $R(T)$.

To find $N(T)$ we need to find all vectors $< a_1, a_2, a_3 >$ such that

$$T(< a_1, a_2, a_3 >) =< a_1 + a_2, 3a_3 >=< 0,0 >.$$

$$a_1 + a_2 = 0 \quad \Longrightarrow \quad a_1 = -a_2$$

$$3a_3 = 0 \quad \Longrightarrow \quad a_3 = 0$$

So $N(T) = \{< a, -a, 0 >\in \mathbb{R}^3 |\ a \in \mathbb{R}\}$.

$$R(T) = \{T(< a_1, a_2, a_3 >) =< a_1 + a_2, 3a_3 >\in \mathbb{R}^2 |\, a_1, a_2, a_3 \in \mathbb{R}\}.$$

Let $< x, y >$ be any vector in $\mathbb{R}^2$. Let's show that $< x, y >\in R(T)$.

$$< a_1 + a_2, 3a_3 >=< x, y >.$$

So $$a_1 + a_2 = x$$

$$3a_3 = y.$$

In particular if $a_1 = x, \quad a_2 = 0, \quad a_3 = \frac{y}{3}$ then

$$T\left(< x, 0, \frac{y}{3} >\right) =< x, y > \implies R(T) = \mathbb{R}^2.$$

Theorem: Let V and W be vector spaces and $T: V \to W$ be linear. Then $N(T)$ is a subspace of V and $R(T)$ is a subspace of W.

Proof: First we show that $N(T)$ is a subspace of V.

Suppose that $v, w \in N(T)$ and $c \in \mathbb{R}$ then

$$T(v + w) = T(v) + T(w) = 0 + 0 = 0 \implies v + w \in N(T).$$

$$T(cv) = cT(v) = c(0) = 0 \implies cv \in N(T).$$

So $N(T)$ is a subspace of V.

Now we show that $R(T)$ is a subspace of W.

Suppose that $w_1, w_2 \in R(T)$ and $c \in \mathbb{R}$ then there exist $v_1, v_2 \in V$ such that

$T(v_1) = w_1$ and $T(v_2) = w_2$.

Thus we have:

$$T(v_1 + v_2) = T(v_1) + T(v_2) = w_1 + w_2 \implies w_1 + w_2 \in R(T).$$

$$T(cv_1) = cT(v_1) = cw_1 \implies cw_1 \in R(T).$$

So $R(T)$ is a subspace of W.

Theorem: Let V and W be vector spaces and $T: V \to W$ be linear. If $B = \{v_1, \dots, v_n\}$ is a basis for V then

$$R(T) = span(T(B)) = span\{T(v_1), \dots, T(v_n)\}.$$

Proof: $T(v_i) \in R(T)$ for each i.

$R(T)$ is a subspace of W $\implies$ $R(T)$ contains $span\{T(v_1), \dots, T(v_n)\}$.

Now suppose $w \in R(T)$ then $w = T(v)$ for some $v \in V$.

Since $\{v_1, \dots, v_n\}$ is a basis for V, $v = \sum_{i=1}^{n} a_i v_i$ for some $a_1, \dots a_n \in \mathbb{R}$.

Since T is linear we have:

$$w = T(v) = T(\sum_{i=1}^{n} a_i v_i) = \sum_{i=1}^{n} a_i T(v_i) \in span(T(B)).$$

So $R(T)$ is contained in the $span\{T(v_1), \dots, T(v_n)\}$.

Since $R(T) \supseteq span\{T(v_1), \dots, T(v_n)\}$ and $R(T) \subseteq span\{T(v_1), \dots, T(v_n)\}$,

$$\implies R(T) = span(T(B)) = span\{T(v_1), \dots, T(v_n)\}.$$

Ex. Define the linear transformation $T: P_2(\mathbb{R}) \to M_{2\times 2}(\mathbb{R})$ by

$$T(f(x)) = \begin{bmatrix} f(3) - f(1) & 0 \\ 0 & f(0) \end{bmatrix}.$$

Find a basis for $R(T)$ and $dimR(T)$.

Since $B = \{1, x, x^2\}$ is a basis for $P_2(\mathbb{R})$

$$\begin{aligned} R(T) &= span\{T(1), T(x), T(x^2)\} \\ &= span\{\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 8 & 0 \\ 0 & 0 \end{bmatrix}\} \\ &= span\{\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix}\}. \end{aligned}$$

Since $\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix}$ are linearly independent (one is not a nonzero multiple of the other) they form a basis for $R(T)$. Hence $dimR(T) = 2$.

Def. Let V and W be vector spaces and $T: V \to W$ be linear. If $N(T)$ and $R(T)$ are finite dimensional, then we define the nullity of , $\boldsymbol{Nullity(T)} = \boldsymbol{dimN(T)}$, and the rank of T, $\boldsymbol{Rank(T)} = \boldsymbol{dimR(T)}$.

Theorem (dimension theorem): Let V and W be vector spaces and $T: V \to W$ be linear. If V is a finite dimensional vector space then

$$Nullity(T) + Rank(T) = \dim(V).$$

Proof: Suppose that $\dim(V) = n$,

$dimN(T) = k$ and $\{v_1, \dots v_k\}$ is a basis for $N(T)$.

Extend $\{v_1, \dots v_k\}$ to a basis $\{v_1, \dots v_n\}$ of V.

Claim: $S = \{T(v_{k+1}), \dots, T(v_n)\}$ is a basis for $R(T)$.

Since $T(v_1) = T(v_2) = \cdots = T(v_k) = 0$, we know from the previous theorem that S generates $R(T)$ since

$$R(T) = span\{T(v_1), \dots, T(v_n)\} = span\{T(v_{k+1}), \dots, T(v_n)\}.$$

Now let's show that S is linearly independent. Suppose

$$b_{k+1}T(v_{k+1}) + \cdots + b_nT(v_n) = 0.$$

Since T is linear: $T(b_{k+1}(v_{k+1}) + \cdots + b_n(v_n)) = 0$.

So $b_{k+1}(v_{k+1}) + \cdots + b_n(v_n) \in N(T)$.

Since $\{v_1, \dots v_k\}$ is a basis for $N(T)$ there exist $c_1, \dots, c_k \in \mathbb{R}$ such that

$$c_1v_1 + \cdots + c_kv_k = b_{k+1}(v_{k+1}) + \cdots + b_n(v_n)$$

$$c_1v_1 + \cdots + c_kv_k - b_{k+1}(v_{k+1}) - \cdots - b_n(v_n) = 0.$$

But $\{v_1, \dots v_n\}$ are linearly independent so $b_{k+1}, \dots, b_n = 0$.

Hence S is linearly independent and a basis for $R(T)$.

Thus $Rank(T) = n - k$ and $Nullity(T) + Rank(T) = \dim(V)$.

Ex. Let $T: \mathbb{R}^3 \to \mathbb{R}^2$ by $T(< a_1, a_2, a_3 >) =< a_1 + a_2, 3a_3 >$. Find the $\dim(R(T))$.

We found earlier that

$$N(T) = \{< a, -a, 0 >| \; a \in \mathbb{R}\} = \{a < 1, -1, 0 > | \; a \in \mathbb{R}\}.$$

So $N(T)$ has a basis of $< 1, -1, 0 >$ and $\dim(N(T)) = 1$.

By the previous theorem we know that $\dim(R(T)) = 2$ since $\dim(\mathbb{R}^3) = 3$.

$$\dim(N(T)) + \dim(R(T)) = \dim(\mathbb{R}^3)$$

$$1 \quad + \; \dim(R(T)) = 3$$

$$\Longrightarrow \quad \dim(R(T)) = 2.$$

Def. Let V and W be vector spaces and $T: V \to W$ be linear. T is called **one-to-one** if $T(v_1) = T(v_2)$ implies $v_1 = v_2$. T is called **onto** if given any $w \in W$ there exists at least one $v \in V$ such that $T(v) = w$.

Theorem: Let V and W be vector spaces and $T: V \to W$ be linear. Then T is one-to-one if and only if $N(T) = \{0\}$.

Proof: Suppose T is one-to-one and $v \in N(T)$.

Then $T(v) = 0 = T(0)$.

But T is one-to-one so $v = 0$.

Now suppose that $N(T) = \{0\}$ and $T(x) = T(y)$.

Then $0 = T(x) - T(y) = T(x - y)$, so $x - y \in N(T)$.

Thus $x - y = 0$ and $x = y$. Thus T is one-to-one.

Theorem: Let V and W be vector spaces of equal (finite) dimension, and let $T: V \to W$ be linear. Then the following are equivalent:

a. T is one-to-one

b. T is onto

c. $Rank(T) = \dim(V)$.

Proof: Recall that $Nullity(T) + Rank(T) = \dim(V)$.

T is one-to-one $\Leftrightarrow N(T) = \{0\}$.

$N(T) = \{0\} \Leftrightarrow Nullity(T) = 0$.

$Nullity(T) = 0 \Leftrightarrow Rank(T) = \dim(V)$.

$Rank(T) = \dim(V) \Leftrightarrow Rank(T) = \dim(W)$.

$Rank(T) = \dim(W) \Leftrightarrow R(T) = W$, i.e. T is onto.

Note: The previous theorem does not hold if V and W are infinite dimensional.

For example, let $V = W = P(\mathbb{R})$ and $T: P(\mathbb{R}) \to P(\mathbb{R})$ by

1. $T(f(x)) = \int_0^x f(t)dt$.

T is one-to-one because $T(f(x)) = T(g(x))$ means

$$\int_0^x f(t)dt = \int_0^x g(t)dt \text{ for all } x.$$

$$\Rightarrow \int_0^x (f(t) - g(t))dt = 0 \text{ for all } x.$$

$$\Rightarrow f(x) = g(x).$$

However, T is not onto as $T(f(x)) \neq$constant function.

2. $T\big(f(x)\big) = f'(x)$.

T is not one-to-one because $T\big(f(x)\big) = T(g(x))$ means

$$f'(x) = g'(x) \implies f(x) = g(x) + C \text{ for any constant } C.$$

However, T is onto because given any $g(x) = a_0 + a_1x + \cdots + a_nx^n \in P(\mathbb{R})$ then $f(x) = \int g(x)dx = a_0x + \frac{1}{2}a_1x^2 + \cdots + \frac{a_n}{n}x^{n+1} + C$ has the property that $T\big(f(x)\big) = g(x)$.

Ex. Let $T: P_2(\mathbb{R}) \to P_2(\mathbb{R})$ by $T\big(f(x)\big) = xf'(x)$. Show that T is a linear transformation. Determine if T is one-to-one and/or onto.

To show that T is a linear transformation, let $f(x), g(x) \in P_2(\mathbb{R})$ and $c \in \mathbb{R}$. Then we have:

$$\begin{aligned} T\big(cf(x) + g(x)\big) &= x[cf(x) + g(x)]' \\ &= x[cf'(x) + g'(x)] \\ &= cxf'(x) + xg'(x) \\ &= cT\big(f(x)\big) + T(g(x)). \end{aligned}$$

Thus T is linear.

T is one-to-one $\Leftrightarrow$ $N(T) = \{0\}$.

$$T\big(f(x)\big) = 0$$

$$xf'(x) = 0$$

$$\implies \quad x = 0 \text{ or } f'(x) = 0.$$

But $f'(x) = 0 \implies f(x) = constant.$

Thus all constant functions $f(x) \in N(T)$.

Hence $N(T) \neq \{0\}$ and T is not one-to-one.

In fact $N(T)$ is spanned by $f(x) = 1$ hence

$\dim\big(N(T)\big) = 1.$

Since $Nullity(T) + Rank(T) = \dim(P_2(\mathbb{R})) = 3$

$Rank(T) = 2$ so $R(T) \neq P_2(\mathbb{R})$ and T is not onto.

$R(T)$ is spanned by $T(1), T(x), T(x^2)$ because $\{1,\ x,\ x^2\}$ is a basis for $P_2(\mathbb{R})$.

$$R(T) = span\{T(1), T(x), T(x^2)\}.$$

$T(1) = x(0) = 0$ since if $f(x) = 1,\ f'(x) = 0.$

$T(x) = x(1) = x$ since if $f(x) = x,\ f'(x) = 1.$

$T(x^2) = x(2x) = 2x^2$ since if $f(x) = x^2,\ f'(x) = 2x.$

Thus $R(T) = span\{0, x, 2x^2\} = span\{\, x, 2x^2\}$

$$= \{p(x) \in P_2(\mathbb{R}) |\ p(x) = ax + 2bx^2,\ \ a, b \in \mathbb{R}\}$$

Since x and $2x^2$ are linearly independent, $dimR(T) = 2.$

Ex. Suppose that $T: \mathbb{R}^2 \to \mathbb{R}^2$ is linear and $T(< 0,1 >) =< 2,3 >$ and $T(< 2,-1 >) =< 1,2 >$. Is T one-to-one?

By the previous theorem with $V = W = \mathbb{R}^2$, we have $dimV = dimW = 2.$

T is one-to-one if T is onto (i.e. $R(T) = \mathbb{R}^2$).

But since $< 0,1 >$ and $< 2,-1 >$ are linearly independent (one is not a multiple of the other), they form a basis for $V = \mathbb{R}^2$. Thus we have:

$$R(T) = span\{T(< 0,1 >),\ T(< 2,-1 >)\} = span\{< 2,3 >, < 1,2 >\}.$$

But $< 2,3 >$ and $<1,2 >$ are also linearly independent and thus a basis for $W = \mathbb{R}^2$.

Hence $R(T) = \mathbb{R}^2$ and T is onto $\Longrightarrow$ T is one-to-one

Ex. Let V and W be vector spaces of equal (finite) dimension, and let $T: V \to W$ be linear. Show that if $\dim(V) > \dim(W)$, then T can't be one-to-one.

$$Nullity(T) + Rank(T) = \dim(V) > \dim\ (W).$$

$R(T)$ is a subspace of W so $\dim\big(R(T)\big) \leq \dim(W)$.

Thus $\dim\big(N(T)\big) + Rank(T) > \dim(W) \ \Longrightarrow\ \dim\ (N(T)) \geq 1$.

Hence $N(T) \neq \{0\}$ and T is not one-to-one.

Theorem: Let V and W be vector spaces and suppose that $\{v_1, \dots, v_n\}$ is a basis for V. For $w_1, \dots, w_n \in W$, there exists exactly one linear transformation $T: V \to W$ such that $T(v_i) = w_i$.

Proof: Given any $v \in V$, $v = a_1 v_1 + \cdots a_n v_n$, where $a_1, \dots, a_n$ are unique.

Define $T: V \to W$ by $T(v) = a_1 w_1 + \cdots + a_n w_n$.

Notice that T is linear since if $u, s \in V$, $d \in \mathbb{R}$ we have:

$u = b_1 v_1 + \cdots b_n v_n$ and $s = c_1 v_1 + \cdots c_n v_n$ so we get:

$du + s = (db_1 + c_1)v_1 + \cdots + (db_n + c_n)v_n$ and

$$\begin{aligned} T(du + s) &= (db_1 + c_1)T(v_1) + \cdots + (db_n + c_n)T(v_n) \\ &= (db_1 + c_1)w_1 + \cdots + (db_n + c_n)w_n \\ &= d(b_1 w_1 + \cdots + b_n w_n) + (c_1 w_1 + \cdots . + c_n w_n) \\ &= dT(u) + T(s). \end{aligned}$$

T is unique because if $U: V \to W$ is a linear transformation with $U(v_i) = w_i$ then

$$\begin{aligned} U(v) &= a_1 U(v_1) + \cdots + a_n U(v_n) \\ &= a_1 w_1 + \cdots \quad + a_n w_n \\ &= T(v). \qquad \text{(Since } T(v_i) = w_i\text{)} \end{aligned}$$

Hence $U = T$.

Corollary: Let V and W be vector spaces and suppose that $\{v_1, \dots, v_n\}$ is a basis for V. If $U, T: V \to W$ are linear transformations with $U(v_i) = T(v_i)$ for $i = 1, \dots, n$, then $U = T$.

In other words, a linear transformation is defined by what it does to a set of basis vectors.