Linear Transformations- HW Problems

In problems 1-4 determine which mappings are linear transformations.

1. $T: \mathbb{R}^2 \to \mathbb{R}^2$ by $T(x_1, x_2) = (x_2,\ x_1x_2)$
2. $T: \mathbb{R}^2 \to \mathbb{R}^3$ by $T(x_1, x_2) = (x_1,\ x_2,\ x_1 + 3x_2)$
3. $T: M_{n\times n}(\mathbb{R}) \to M_{n\times n}(\mathbb{R})$ by $T(A) = A + I$, $I =$identity matrix
4. $T: C(\mathbb{R}) \to \mathbb{R}$ by $T(f) = f(3)$.

In problems 5-7 T is a linear transformation. Find a basis for $\ker(T)$ and $Im(T)$ (ie $R(T)$). Also determine if T is one-to-one and/or onto.

5. $T: \mathbb{R}^2 \to \mathbb{R}^3$ by $T(x_1, x_2) = (0,\ x_1 + x_2,\ 3x_2 - x_1)$
6. $T: P_2(\mathbb{R}) \to P_2(\mathbb{R})$ by $T(p(x)) = xp'(x)$,
   where $p(x) = a_0 + a_1x + a_2x^2$
7. $T: M_{2\times2}(\mathbb{R}) \to M_{2\times2}(\mathbb{R})$ by

$$T\left(\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}\right) = \begin{bmatrix} a_{11} - a_{12} & 0 \\ 0 & a_{22} - a_{21} \end{bmatrix}$$

8. Let $T: \mathbb{R}^2 \to \mathbb{R}$ be a linear transformation with $T(2,1) = -2$ and $T(1,-3) = 3$. Find $T(5,6)$.

9. Does there exist a linear transformation $T: \mathbb{R}^2 \to \mathbb{R}^3$ such that $T(-3,2) = (4, 3, 2)$ and $T(6,-4) = (-4,-3,-2)$?

10. Let V and W be finite dimensional vector spaces. Suppose $T: V \to W$ is a linear transformation. Prove that if

a. $\dim(V) < dim(W)$, then T can't be onto

b. $\dim(V) > dim(W)$, then T can't be one-to-one

Hint: Nullity(T) +Rank$(T) = \dim(V)$.

11. Suppose that U, V, and W are vector spaces and $T_1: U \to V$ and $T_2: V \to W$ are linear transformations. Prove that $T_2 \circ T_1: U \to W$ defined by $T_2 \circ T_1(u) = T_2(T_1(u))$ is a linear transformation.

12. Let V and W be finite dimensional vector spaces and $T: V \to W$ a linear transformation. Suppose that $\dim(V) = \dim(W)$ and $N(T) = \{0\}$. Prove that T is onto.