Linear Independence

Given a set of vectors $v_1, \dots v_n$ in a vector space V, we know that:

Span($v_1, \dots, v_n$) = $\{\alpha_1 v_1 + \cdots + \alpha_n v_n,\ \ \alpha_i \in \mathbb{R}\} = W$ is a subspace of V.

However, it might not be necessary to have all of the vectors $v_1, v_2, \dots, v_n$ to span W. It's often useful to know the minimum number of vectors needed to span a vector space.

Ex. Let $v_1 = <1, 0, 0>$, $v_2 = <0, 1, 0>$, $v_3 = <2, 5, 0>$. Show that the

Span $\{v_1, v_2, v_3\}$ $=$ Span $\{v_1, v_2\}$.

Notice that we can write v_3 as a linear combination of v_1 and v_2:

$$v_3 = 2v_1 + 5v_2$$

Thus any vector w that can be created as a linear combination of v_1, v_2, v_3

$$w = \alpha_1 v_1 + \alpha_2 v_2 + \alpha_3 v_3$$

can also be created as a linear combination of just v_1 and v_2:

$$w = \beta_1 v_1 + \beta_2 v_2.$$

For example, let's say:

$$w = 3v_1 + 2v_2 + 4v_3.$$

Since we know $v_3 = 2\, v_1 + 5\, v_2$, we have:

$$w = 3v_1 + 2v_2 + 4v_3 = 3v_1 + 2v_2 + 4(2\, v_1 + 5\, v_2)$$

$$w = 3v_1 + 2v_2 + 8v_1 + 20\, v_2 = 11\, v_1 + 22\, v_2.$$

Whenever one vector, w, can be written as a linear combination of other vectors, $v_1, v_2, \dots, v_n$, we say the vectors $v_1, v_2, \dots, v_n, w$ are **linearly dependent**.

A set of vectors $v_1, \dots, v_n$ are said to be **linearly independent** if:

$$c_1 v_1 + c_2 v_2 + \dots + c_n v_n = 0$$

implies that: $c_1 = c_2 = c_3 = \cdots = c_n = 0.$

Notice that if a set of vectors $v_1, v_2, \dots, v_n$ are linearly dependent, at least one vector can be written as a linear combination of the other vectors.

Let's assume that v_1 can be written as a linear combination of $v_2, \dots, v_n$, that is:

$$v_1 = c_2 v_2 + c_3 v_3 + \cdots + c_n v_n \text{ ; where not all of the } c_i's \text{ are } 0.$$

But that means that:

$$c_2 v_2 + c_3 v_3 + \cdots + c_n v_n - v_1 = 0; \text{ where not all of the } c_i's \text{ are } 0.$$

Hence if the set of vectors $v_1, v_2, \dots, v_n$ are linearly dependent, then they can't be linearly independent.

Conversely, if a set of vectors is linearly independent then they can't be linearly dependent since if they were linearly dependent we could find a set of $c_i's$, not all 0 such that $c_1 v_1 + c_2 v_2 + \cdots + c_n v_n = 0$.

If $v_1, v_2, \dots, v_n$ are linearly dependent, then some vector in this set, say v_i can be written as a linear combination of the other vectors. Thus

$$span\{v_1, v_2, \dots, v_{i-1}, v_{i+1}, \dots, v_n\} = span\{v_1, v_2, \dots, v_n\}.$$

That is, including v_i in the spanning set doesn't increase the span of the vectors.

Thus the minimal set of vectors that spans a given finite dimensional vector space generated by $v_1, \dots, v_n$ must be linearly independent.

Ex. Show that the vectors $v_1 =< 1, 1, 0 >$, $v_2 =< 1, 0, 1 >$, and $v_3 =< 0,1,1 >$ are linearly independent in the vector space $\mathbb{R}^3$ (we will always assume that the vector space $\mathbb{R}^n$ has the standard vector addition and scalar multiplication unless otherwise stated).

We work from the definition of linearly independence. So assume the there are $c_i's$ such that

$$c_1 v_1 + c_2 v_2 + c_3 v_3 = 0; \qquad c_i \in \mathbb{R}$$

and let's show that $c_1 = c_2 = c_3 = 0$.

$$c_1 < 1, 1, 0 > + c_2 < 1, 0, 1 > + c_3 < 0, 1, 1 > = < 0,0,0 >$$

$$< c_1 + c_2,\ \ c_1 + c_3,\ \ c_2 + c_3 > = < 0,0,0 >.$$

$$\Longrightarrow \qquad \begin{aligned} c_1 + c_2 \qquad &= 0 \\ c_1 + \qquad c_3 &= 0 \\ c_2 + c_3 &= 0. \end{aligned}$$

Subtracting equation one from equation two we get:

$$\begin{aligned} c_1 + c_2 \qquad &= 0 \\ -c_2 + c_3 &= 0 \\ c_2 + c_3 &= 0. \end{aligned}$$

By adding equation two to equation three we see that $c_3 = 0$.

From equation two we see that if $c_3 = 0$ then $c_2 = 0$.

From equation one we see that if $c_2 = 0$ then $c_1 = 0$.

Hence $c_1 = c_2 = c_3 = 0$ and the three vectors are linearly independent.

Ex. Show the vectors $v_1 =< 2, 1, 1 >$, $v_2 =< -1, 2, 1 >$, and $v_3 =< 1,8,5 >$ are linearly dependent in $\mathbb{R}^3$.

We must show that we can find c_1, c_2, c_3, not all zero such that:

$$c_1 v_1 + c_2 v_2 + c_3 v_3 = 0; \qquad c_i \in \mathbb{R}\,.$$

$$c_1 < 2, 1, 1 > + c_2 < -1, 2, 1 > + c_3 < 1,8,5 > = < 0,0,0 >$$

$$< 2c_1 - c_2 + c_3,\ (c_1 + 2c_2 + 8c_3),\ c_1 + c_2 + 5c_3 > = < 0,0,0 >.$$

Equating the components we get:

$$2c_1 - c_2 + c_3 = 0$$

$$c_1 + 2c_2 + 8c_3 = 0$$

$$c_1 + c_2 + 5c_3 = 0.$$

Let's start by rearranging the order of the equations so we have c_1 with a coefficient of 1.

$$c_1 + 2c_2 + 8c_3 = 0$$

$$c_1 + \ \ c_2 + 5c_3 = 0$$

$$2c_1 - \ \ c_2 + \ \ c_3 = 0.$$

Subtract equation one from equation two and subtract 2 times equation one from equation three:

$$\begin{aligned} c_1 + 2c_2 + \ 8c_3 &= 0 \\ -c_2 - \ 3c_3 &= 0 \\ -5c_2 - 15c_3 &= 0. \end{aligned}$$

Multiply equation two by -1:

$$\begin{aligned} c_1 + 2c_2 + \ 8c_3 &= 0 \\ c_2 + \ 3c_3 &= 0 \\ -5c_2 - 15c_3 &= 0. \end{aligned}$$

Subtract 2 times equation two from equation one and add 5 times equation two to equation three:

$$\begin{aligned} c_1 + \qquad 2c_3 &= 0 \\ c_2 + 3c_3 &= 0 \\ 0 &= 0. \end{aligned}$$

Now solve for c_1 and c_2 in terms of c_3:

$$\begin{aligned} c_1 &= -2c_3 \\ c_2 &= -3c_3. \end{aligned}$$

Thus for any real number c_3, we can choose $c_1 = -2c_3$ and $c_2 = -3c_3$.

For example, if $c_3 = 1$ then $c_1 = -2$ and $c_2 = -3$ and

$$-2v_1 - 3v_2 + v_3 = < 0,0,0 >;$$

$$-2 < 2,1,1 > -3 < -1,2,1 > + < 1,8,5 > = < 0,0,0 >.$$

If instead we had chosen $c_3 = -2$, then $c_1 = 4$ and $c_2 = 6$ and

$$4 < 2,1,1 > +6 < -1,2,1 > -2 < 1,8,5 > = < 0,0,0 >.$$

Hence the vectors $v_1 = < 2,1,1 >$, $v_2 = < -1,2,1 >$, and $v_3 = < 1,8,5 >$ are linearly dependent in $\mathbb{R}^3$

Ex. Determine whether the following vectors are linearly independent in $P_2(\mathbb{R})$:

a. $x^2 + 1, \quad x + 1, \quad x^2 + x.$

b. $2x^2 + x + 1, \quad -x^2 + 2x + 1, \quad x^2 + 8x + 5.$

a. $$c_1(x^2 + 1) + c_2(x + 1) + c_3(x^2 + x) = 0$$
$$(c_1 + c_3)x^2 + (c_2 + c_3)x + (c_1 + c_2) = 0$$

So we must have the coefficients of x^2, x and the constant term equal 0:

$$\begin{aligned} c_1 + \quad\;\; c_3 &= 0 \\ c_2 + c_3 &= 0 \\ c_1 + c_2 \quad\;\; &= 0. \end{aligned}$$

Notice these are the same equations we got when we showed the vectors $< 1,1,0 >, < 1,0,1 >$, and $< 0,1,1 >$ were linearly independent in $\mathbb{R}^3$ (it's no accident that these are the same equations). Thus $c_1 = c_2 = c_3 = 0$ and $x^2 + 1$, $x + 1$, and $x^2 + x$ are linearly independent in $P_2(\mathbb{R})$.

b. $$c_1(2x^2+x+1)+c_2(-x^2+2x+1)+c_3(x^2+8x+5)=0$$
$$(2c_1-c_2+c_3)x^2+(c_1+2c_2+8c_3)x+(c_1+c_2+5c_3)=0.$$

Equating the coeffiecients of x^2, x, and the constant term we have:

$$2c_1-\ c_2+\ c_3=0$$
$$c_1+2c_2+8c_3=0$$
$$c_1+\ c_2+\ 5c_3=0.$$

But these are the equations we got when we showed that the vectors $< 2,1,1 >$, $< -1,2,1 >$ and $< 1,8,5 >$ were linearly dependent in $\mathbb{R}^3$.

Thus for any real number c_3, if we let $c_1=-2c_3$ and $c_2=-3c_3$ then

$$c_1(2x^2+x+1)+c_2(-x^2+2x+1)+c_3(x^2+8x+5)=0.$$

For example, if $c_3=1$, then $c_1=-2$ and $c_2=-3$ and

$$-2(2x^2+x+1)-3(-x^2+2x+1)+1(x^2+8x+5)=0.$$

Thus the polynomials $2x^2+x+1$, $-x^2+2x+1$, x^2+8x+5 are linearly dependent in $P_2(\mathbb{R})$.

Ex. Determine whether the following vectors are linearly independent in $M_{2\times2}(\mathbb{R})$:

a. $\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}, \begin{bmatrix}0 & 0\\0 & 1\end{bmatrix}, \begin{bmatrix}0 & 0\\2 & 1\end{bmatrix}$

b. $\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}, \begin{bmatrix}0 & 0\\1 & 0\end{bmatrix}, \begin{bmatrix}3 & 0\\4 & 3\end{bmatrix}$

a. $c_1\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix} + c_2\begin{bmatrix}0 & 0\\0 & 1\end{bmatrix} + c_3\begin{bmatrix}0 & 0\\2 & 1\end{bmatrix} = \begin{bmatrix}0 & 0\\0 & 0\end{bmatrix}$

$$\begin{bmatrix}c_1 & 0\\0 & c_1\end{bmatrix} + \begin{bmatrix}0 & 0\\0 & c_2\end{bmatrix} + \begin{bmatrix}0 & 0\\2c_3 & c_3\end{bmatrix} = \begin{bmatrix}0 & 0\\0 & 0\end{bmatrix}$$

$$\begin{bmatrix}c_1 & 0\\2c_3 & c_1 + c_2 + c_3\end{bmatrix} = \begin{bmatrix}0 & 0\\0 & 0\end{bmatrix}.$$

Equating the entries we get:

$$\begin{aligned} c_1 \quad\quad\quad &= 0\\ 2c_3 &= 0\\ c_1 + c_2 + c_3 &= 0. \end{aligned}$$

The only solution is $c_1 = c_2 = c_3 = 0$, so the vectors are linearly independent.

b. $c_1\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix} + c_2\begin{bmatrix}0 & 0\\1 & 0\end{bmatrix} + c_3\begin{bmatrix}3 & 0\\4 & 3\end{bmatrix} = \begin{bmatrix}0 & 0\\0 & 0\end{bmatrix}$

$$\begin{bmatrix}c_1 & 0\\0 & c_1\end{bmatrix} + \begin{bmatrix}0 & 0\\c_2 & 0\end{bmatrix} + \begin{bmatrix}3c_3 & 0\\4c_3 & 3c_3\end{bmatrix} = \begin{bmatrix}0 & 0\\0 & 0\end{bmatrix}$$

$$\begin{bmatrix}c_1 + 3c_3 & 0\\c_2 + 4c_3 & c_1 + 3c_3\end{bmatrix} = \begin{bmatrix}0 & 0\\0 & 0\end{bmatrix}.$$

Equating the entries we get:

$$\begin{aligned} c_1 + \quad 3c_3 &= 0\\ c_2 + 4c_3 &= 0\\ c_1 + \quad 3c_3 &= 0. \end{aligned}$$

Subtracting equation one from equation three we get:

$$\begin{aligned} c_1 + \quad 3c_3 &= 0\\ c_2 + 4c_3 &= 0\\ 0 &= 0. \end{aligned}$$

Thus for any value of c_3, $c_1 = -3c_3$ and $c_2 = -4c_3$ and

$$-3c_3\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix} - 4c_3\begin{bmatrix}0 & 0\\1 & 0\end{bmatrix} + c_3\begin{bmatrix}3 & 0\\4 & 3\end{bmatrix} = \begin{bmatrix}0 & 0\\0 & 0\end{bmatrix}.$$

Thus $\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}$, $\begin{bmatrix}0 & 0\\1 & 0\end{bmatrix}$, $\begin{bmatrix}3 & 0\\4 & 3\end{bmatrix}$ are linearly dependent in $M_{2\times 2}(\mathbb{R})$.

Theorem: Let V be a vector space, and let $S_1 \subseteq S_2 \subseteq V$. If S_1 is linearly dependent then so is S_2 .

Proof: By definition of linear dependence there exist real numbers $c_1, \dots, c_n$, not all zero such that $c_1 v_1 + c_2 v_2 + \dots + c_n v_n = 0$ for $S_1 = \{v_1, \dots, v_n\}$.

Suppose $S_2 = \{v_1, \dots, v_n, v_{n+1}, \dots, v_{n+m}\}$.

If we take $c_{n+1}, \dots, c_{n+m} = 0$ then

$$
\begin{aligned}
c_1 v_1 + c_2 v_2 + \dots + c_n v_n + c_{n+1} v_{n+1} + \cdots + c_{n+m} v_{n+m} \\
&= c_1 v_1 + c_2 v_2 + \dots + c_n v_n \\
&= 0
\end{aligned}
$$

where $c_1, \dots, c_{n+m}$ are not all zero.

Thus S_2 is linearly dependent.

Corollary: Let V be a vector space, and let $S_1 \subseteq S_2 \subseteq V$. If S_2 is linearly independent, then S_1 is linearly independent.

Proof: Suppose that S_1 is linearly dependent.

Then by the previous theorem S_2 must be linearly dependent, which is a contradiction. Thus S_1 is linearly independent.

Theorem: Let S be a linearly independent subset of a vector space V, and let $v \in V$ that is not in S. Then $S \cup \{v\}$ is linearly dependent if and only if $v \in span(S)$.

Proof: If $S \cup \{v\}$ is dependent then there are vectors $v_1, \ldots, v_n \in S \cup \{v\}$ such that: $a_1 v_1 + \cdots + a_n v_n = 0$ for nonzero real numbers $a_1, \ldots, a_n$.

Since S is linearly independent, one of the $v_i{}'s$ must be v.

Let's say $v_1 = v$, then

$$a_1 v + \cdots + a_n v_n = 0 \quad \text{and}$$

$$v = -\frac{1}{a_1}(a_2 v_2 + \cdots + a_n v_n).$$

Thus $v \in span(S)$.

Now assume that $v \in span(S)$ and let's show that $S \cup \{v\}$ is linearly dependent.

Since $v \in span(S)$ there exist vectors $v_1, \ldots, v_m \in S$ such that

$$v = b_1 v_1 + \cdots + b_m v_m.$$

Hence we have:

$$0 = b_1 v_1 + \cdots + b_m v_m - v.$$

Since $v \neq v_i$ for $i = 1, \ldots, m$ the coefficient of v on the RHS is nonzero, So the set $\{v_1, \ldots, v_n, v\}$ is linearly dependent.

Since $\{v_1, \ldots, v_n, v\} \subseteq S \cup \{v\}$, $\quad S \cup \{v\}$ is linearly dependent.