Linear Systems and Linear Combinations

Def. Let V be a vector space and S a nonempty subset of V. A vector $v \in V$ is called a **linear combination** of vectors in S if there exists a finite number of vectors $v_1, v_2, \dots, v_n \in S$ such that:

$$v = a_1 v_1 + a_2 v_2 + \cdots + a_n v_n.$$

In this case we say v is a linear combination of $v_1, v_2, \dots, v_n$ and $a_1, a_2, \dots, a_n$ are called the **coefficients** of the linear combination.

When working with vector spaces it's very common to ask given a vector v is it a linear combination of a given set of vectors $v_1, v_2, \dots, v_n$. Thus it's useful to be able to find an answer to this question. We'll demonstrate this method through an example.

Ex. Let $v_1 =< 1, 3, 2 >$, $v_2 =< -4, -12, -8 >$, $v_3 =< -1, 0, 4 >$, and $v_4 =< 1, -3, -10 >$ be vectors in $\mathbb{R}^3$. Is the vector in $\mathbb{R}^3$ given by $< 3, 12, 12 >$ a linear combination of v_1, v_2, v_3, and v_4? If so, find a set of coefficients a_1, a_2, a_3, and a_4 such that :

$$< 3, 12, 12 >= a_1 v_1 + a_2 v_2 + a_3 v_3 + a_4 v_4.$$

Let's start by assuming we can find $a_1, a_2, a_3, a_4 \in \mathbb{R}$ such that

$$< 3, 12, 12 >= a_1 v_1 + a_2 v_2 + a_3 v_3 + a_4 v_4$$

$$= a_1 < 1,3,2 > + a_2 < -4, -12, -8 > + a_3 < -1,0,4 > + a_4 < 1, -3, -10 >$$

$$=< a_1 - 4a_2 - a_3 + a_4,\ (3a_1 - 12a_2 - 3a_4),\ 2a_1 - 8a_2 + 4a_3 - 10a_4 >.$$

Since the corresponding components on each side of the equation must be equal, we have the following simultaneous equations:

$$\begin{aligned} a_1 - 4a_2 - a_3 + a_4 &= 3 \\ 3a_1 - 12a_2 - \qquad 3a_4 &= 12 \\ 2a_1 - 8a_2 + 4a_3 - 10a_4 &= 12. \end{aligned}$$

Our goal is to see if we can find $a_1, a_2, a_3, a_4 \in \mathbb{R}$ that will satisfy all three equations.

We start by adding or subtracting multiples of these equations so that the coefficient of a_1 in the first equation is one and the subsequent equations do NOT contain an a_1 term. (Note: if the first equation doesn't have an a_1 term you can switch the order of the equations so that the first equation does have an a_1 term. At least one of the three equations must have an a_1 term.).

Notice that our first equation already has an a_1 term with a coefficient of one (if the non-zero coeffiecient is not one, divide the equation by the coefficient or switch the order of the equations with one that does have a coefficient of one.). However, we must eliminate the a_1 term from equations two and three. We can eliminate the a_1 term from equation two by multiplying equation one by -3 and adding it to equation two.

$$\begin{aligned} -3a_1 + 12a_2 + 3a_3 - 3a_4 &= -9 \\ \underline{3a_1 - 12a_2 + \qquad -3a_4} &\underline{= 12} \\ 3a_3 - 6a_4 &= 3 \end{aligned}$$

(new equation two).

To eliminate the a_1 term from equation three, multiply equation one by -2 and add it to equation three.

$$\begin{array}{r} -2a_1 + 8a_2 + 2a_3 - \ \ 2a_4 = -6 \\ \underline{2a_1 - 8a_2 + 4a_3 - 10a_4 = 12} \\ 6a_3 - 12a_4 = 6 \end{array}$$ (new equation three).

So now we have the following three equations:

$$\begin{aligned} a_1 - 4a_2 - \quad a_3 + \quad a_4 &= 3 \\ 3a_3 - \ \ 6a_4 &= 3 \\ 6a_3 - 12a_4 &= 6. \end{aligned}$$

Notice that a_2 disappeared from equations two and three (this does not happen in general). Now we want to get the next highest index of a, in this case a_3, to have a coefficient of 1 in the second equation and not appear in the third equation. Once again, we can switch the order of equations two and three if the next highest index of a only apears in equation three.

So we start by dividing equation two by 3 to make the coefficient of a_3 be 1.

$$\begin{aligned} a_1 - 4a_2 - \quad a_3 + \quad a_4 &= 3 \\ a_3 - \ \ 2a_4 &= 1 \\ 6a_3 - 12a_4 &= 6. \end{aligned}$$

Now multiply equation two by -6 and add it to equation three.

$$\begin{array}{r} -6a_3 + 12a_4 = -6 \\ \underline{6a_3 - 12a_4 \ = 6} \\ 0 = 0. \end{array}$$ (equation 3 now drops out).

So now our equations are:

$$a_1 - 4a_2 - \quad a_3 + \quad a_4 = 3$$
$$a_3 - \ 2a_4 = 1$$

Now that the coefficient of each of the leading terms in each equation is one, we now want to eliminate that leading term in all of the equations above it (there is no equation above the first equation).

So now let's eliminate a_3 in equation one by adding equation one to equation two.

$$\begin{array}{r} a_1 - 4a_2 - \quad a_3 + \ a_4 = 3 \\ \underline{\qquad\qquad\qquad\qquad a_3 - 2a_4 = 1} \\ a_1 - 4a_2 - \qquad\qquad a_4 = 4. \end{array}$$

So our equations are now:

$$a_1 - 4a_2 + \qquad -a_4 = 4$$
$$a_3 \quad - 2a_4 = 1 \qquad\qquad (*)$$

Now solve each of the equations for the first unknown in the equation:

$$a_1 = 4 + 4a_2 + a_4$$
$$a_3 = 1 + 2a_4$$

Thus for any choice of a_2 and a_4:

$$a_1 = 4 + 4a_2 + a_4$$

$$a_2 = a_2$$

$$a_3 = 1 + 2a_4$$

$$a_4 = a_4$$

and we have:

$$< 3, 12, 12 >= a_1v_1 + a_2v_2 + a_3v_3 + a_4v_4.$$

For example, if we choose $a_2 = a_4 = 0$ we have $a_1 = 4$, and $a_3 = 1$

$$< 3, 12, 12 > = 4v_1 + v_3$$

$$= 4 < 1, 3, 2 > + < -1, 0, 4 >$$

If we choose $a_2 = -2$ and $a_4 = 1$ then $a_1 = -3$, and $a_3 = 3$ and we get:

$$-3v_1 - 2v_2 + 3v_3 + v_4$$

$$= -3 < 1, 3, 2 > -2 < -4, -12, -8 > +3 < -1, 0, 4 > + < 1, -3, -10 >$$

$$=< 3, 12, 12 >.$$

Remember, the step before we solve for each a_i, exhibited by $(*)$ in this example has:

1. The first nonzero coefficient in each equation is one.
2. If an unknown is the first unknown with a nonzero coefficient in the some equation, then that unknown doesn't appear in other equations.
3. The first unknown to appear in any equation has a larger subscript than the first unknown in any preceding equation.

For example, notice that we can't have:

A.
$$\begin{aligned} a_1 + 2a_2 - 3a_4 &= 6 \\ 2a_3 + 4a_4 &= 11 \end{aligned}$$

because the first unknown in equation two doesn't have a coefficient of 1

B.
$$\begin{aligned} a_1 - 2a_2 + 3a_3 - \ a_4 &= 3 \\ a_2 + 3a_3 - 2a_4 &= 2 \\ a_3 + \ a_4 &= 1 \end{aligned}$$

because a_2 appears in equation one and a_3 appears in equations one and two.

C.
$$\begin{aligned} a_1 + 2a_2 + a_4 - a_5 &= 6 \\ a_3 - \ a_4 + a_5 &= 11 \\ a_2 + a_4 - a_5 &= 3 \end{aligned}$$

because a_2 appears in equation three and has a lower subscript than a_3, the first unknown that appears in equation two and a_2 appears in equation one.

If a vector v is not a linear combination of vectors $v_1, v_2, \dots, v_n$ then when we apply the previous method we will eventually find a "nonsensical" equation of the form $0 = c$, where c is a nonzero number.

Ex. Show that $-2x + 4$ is not a linear combination of $x^2 - 4x + 4$ and $-2x^2 + 6x + 4$.

We start by assuming that $-2x + 4$ is a linear combination of $x^2 - 4x + 4$ and $-2x^2 + 6x + 4$.

$$\begin{aligned} -2x + 4 &= a_1(x^2 - 4x + 4) + a_2(-2x^2 + 6x + 4) \\ &= (a_1 - 2a_2)x^2 + (-4a_1 + 6a_2)x + (4a_1 + 4a_2). \end{aligned}$$

Equating the coefficients on the RHS and LHS we get:

$$\begin{aligned} a_1 - 2a_2 &= 0 \\ -4a_1 + 6a_2 &= -2 \\ 4a_1 + 4a_2 &= 4. \end{aligned}$$

a_1 already appears in the first equation with a coefficient of 1, so we now must eliminate a_1 from equation two and three. We can do this by multiplying equation one by 4 and adding it to equation two and multiplying equation one by -4 and additing it to equation three. This gives us:

$$\begin{aligned} a_1 - 2a_2 &= 0 \\ -2a_2 &= -2 \\ 12a_2 &= 4. \end{aligned}$$

Now divide equation two by -2 to get:

$$a_1 - 2a_2 = 0$$
$$a_2 = 1$$
$$12a_2 = 4.$$

Now multiply equation two by -12 and add it to equation three to get:

$$a_1 - 2a_2 = 0$$
$$a_2 = 1$$
$$0 = -8.$$

Equation three is now a "nonsensical" equation and therefore there is no solution to the system of equations and therefore $-2x + 4$ is not a linear combination of $x^2 - 4x + 4$ and $-2x^2 + 6x + 4$.

Def. Let S be a nonempty subset of a vector space V. The **span of S**, written $span(S)$, is the set of all linear combinations of the vectors in S. We define $span(\phi) = \{0\}$.

Ex. The span of $S = \{< 1, 0, 0 >, < 0, 0, 1 >\}$ in $\mathbb{R}^3$ is the set of vectors of the form

$$span(S) = \{a < 1, 0, 0 > +b < 0, 0, 1 >| \ a, b \in \mathbb{R}\}$$
$$= \{< a, 0, b >| \ a, b \in \mathbb{R}\}.$$

This is the x-z plane in $\mathbb{R}^3$. Notice that the $span(S)$ is a subspace of $\mathbb{R}^3$.

Theorem: The span of any subset S of a vector space V is a subspace of V. In addition, any subspace of V that contains S must also contain the span of S.

Proof: If $S = \phi$ then $span(S) = \{0\}$ (by definition), which is a subspace of any vector space V.

Suppose that $w, u \in span(S)$. Let's show that $w + u \in span(S)$.

Since $w \in span(S)$ there exist $w_1, \dots, w_n \in S$ and $a_1, \dots, a_n \in \mathbb{R}$ such that

$$w = a_1 w_1 + \cdots + a_n w_n.$$

Since $u \in span(S)$ there exist $u_1, \dots, u_m \in S$ and $b_1, \dots, b_m \in \mathbb{R}$ such that

$$u = b_1 u_1 + \cdots + b_m u_m.$$

But then:

$$w + u = a_1 w_1 + \cdots + a_n w_n + b_1 u_1 + \cdots + b_m u_m.$$

Note: Some of the $w_i{}'s$ and $u_j{}'s$ might be the same vector, but then we can just add their coefficients.

So $w + u \in span(S)$.

Now let's show that $cw \in span(S)$ for any $c \in \mathbb{R}$ and $w \in span(S)$.

$$w = a_1 w_1 + \cdots + a_n w_n \implies \text{c}w = ca_1 w_1 + \cdots + ca_n w_n.$$

Thus $cw \in span(S)$ and $span(S)$ is a subspace of V.

Now let's show that if W is a subspace of V and $W \supseteq S$ then $W \supseteq span(S)$.

If $w \in span(S)$ then

$$w = a_1 w_1 + \cdots + a_n w_n; \quad w_1, \ldots, w_n \in S \subseteq W.$$

But if $w_1, \ldots, w_n \in W$ and W is a vector space then

$$w = a_1 w_1 + \cdots + a_n w_n \in W$$

Since the scalar multiple of any vector in W is in W and the sum of any two vectors in W is in W.

Thus $W \supseteq span(S)$.

Def. A subset S of a vector space V **generates or spans *V*** if $span(S) = V$. In this case we say that S generates or spans V.

Ex. Show that the vectors $< 1, 1 >$ and $< 1, -1 >$ generate $\mathbb{R}^2$.

We must show that every vector $< x, y > \in \mathbb{R}^2$ is a linear combination of $< 1, 1 >$ and $< 1, -1 >$

$$a < 1, 1 > + b < 1, -1 > = < x, y >$$

$$< a + b, a - b > = < x, y >.$$

$$\begin{array}{ll} a+b=x & \\ \underline{a-b=y} \quad \text{(add)} & \\ 2a \quad = x+y \implies a=\frac{x+y}{2} & \end{array} \qquad \begin{array}{l} a+b=x \\ \underline{a-b=y} \quad \text{(subtract)} \\ 2b=x-y \implies b=\frac{x-y}{2}. \end{array}$$

For example, if we take $<5,3>\in \mathbb{R}^2$ then:

$$x=5,\ \ y=3 \ \text{ and}$$

$$a=\frac{x+y}{2}=\frac{5+3}{2}=4$$

$$b=\frac{x-y}{2}=\frac{5-3}{2}=1.$$

Thus we have:

$$4<1,1>+1<1,-1>=<5,3>.$$

Ex. Show that $x^2+1,\ x+1,$ and x^2+x generate $P_2(\mathbb{R})$, the vector space of polynomials of degree less than or equal to two.

We must show given any polynomial in $P_2(\mathbb{R})$ we can write it as a linear combination of $x^2+1,\ x+1,$ and x^2+x.

Any polynomial in $P_2(\mathbb{R})$ can be written as $b_0+b_1x+b_2x^2,\ \ b_i\in\mathbb{R}$.

So we must find $a_1, a_2,$ and a_3 such that:

$$a_1(x^2+1)+a_2(x+1)+a_3(x^2+x)=b_0+b_1x+b_2x^2$$

$$(a_1+a_3)x^2+(a_2+a_3)x+(a_1+a_2)=b_0+b_1x+b_2x^2.$$

Equating the coefficients of x^2, x, and the constant term we get:

$$\begin{aligned} a_1 + \quad a_3 &= b_2 \\ a_2 + a_3 &= b_1 \\ a_1 + a_2 \quad &= b_0. \end{aligned}$$

Since the coefficient of a_1 is already 1 and there is no a_1 term in equation two, subtract equation one from equation three to eliminate a_1 from equation three. This gives us:

$$\begin{aligned} a_1 + \quad a_3 &= b_2 \\ a_2 + a_3 &= b_1 \\ a_2 - a_3 &= b_0 - b_2. \end{aligned}$$

Now subtract equation two from equation three to eliminate a_2 in equation three:

$$\begin{aligned} a_1 + \quad a_3 &= b_2 \\ a_2 + a_3 &= b_1 \\ -2a_3 &= b_0 - b_2 - b_1. \end{aligned}$$

Divide equation three by -2 to make the coefficient of a_3 equal to 1.

$$\begin{aligned} a_1 + \quad a_3 &= b_2 \\ a_2 + a_3 &= b_1 \\ a_3 &= \frac{(-b_0+b_2+b_1)}{2}. \end{aligned}$$

To eliminate a_3 from equation one and equation two, just subtract equation three from each of them to get:

$$a_1 = \frac{b_0 - b_1 + b_2}{2}$$

$$a_2 = \frac{b_0 + b_1 - b_2}{2}$$

$$a_3 = \frac{-b_0 + b_1 + b_2}{2}.$$

Thus we have shown that any polynomial in $P_2(\mathbb{R})$ can be written as:

$$b_0 + b_1 x + b_2 x^2 = \left(\frac{b_0 - b_1 + b_2}{2}\right)(x^2 + 1) + \left(\frac{b_0 + b_1 - b_2}{2}\right)(x + 1) + \left(\frac{-b_0 + b_1 + b_2}{2}\right)(x^2 + x).$$

Thus $x^2 + 1,\ x + 1,$ and $x^2 + x$ generate $P_2(\mathbb{R})$.

For example, we can write $-2 + 4x + 2x^2;$ where $b_0 = -2,\ b_1 = 4,\ b_2 = 2$ as

$$-2 + 4x + 2x^2 = \left(\frac{-2-4+2}{2}\right)(x^2 + 1) + \left(\frac{-2+4-2}{2}\right)(x + 1) + (\frac{2+4+2}{2})(x^2 + x)$$
$$= -2(x^2 + 1) + 4(x^2 + x).$$

Ex. Show that $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$, $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, and $\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$ generate $M_{2\times 2}(\mathbb{R})$, 2×2 matrices with real entries.

Any element $B \in M_{2\times 2}(\mathbb{R})$ can be written as :

$$B = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix}, \ b_{ij} \in \mathbb{R}.$$

So we must show that we can find a_1, a_2, a_3, and a_4 such that

$$a_1 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + a_2 \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} + a_3 \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} + a_4 \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix}$$

$$\begin{bmatrix} a_1 + a_2 + a_4 & a_2 + a_3 + a_4 \\ a_3 + a_4 & a_1 + a_2 \end{bmatrix} = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix}.$$

Setting the corresponding entries equal to eachother we get:

$$\begin{aligned} a_1 + a_2 + \quad a_4 &= b_{11} \\ a_2 + a_3 + a_4 &= b_{12} \\ a_3 + a_4 &= b_{21} \\ a_1 + a_2 \quad\quad &= b_{22}. \end{aligned}$$

Start by subtracting equation one from equation four:

$$\begin{aligned} a_1 + a_2 + \quad a_4 &= b_{11} \\ a_2 + a_3 + a_4 &= b_{12} \\ a_3 + a_4 &= b_{21} \\ -a_4 &= b_{22} - b_{11}. \end{aligned}$$

Multiply equation four by -1.

$$\begin{aligned} a_1 + a_2 + \quad\quad a_4 &= b_{11} \\ a_2 + a_3 + a_4 &= b_{12} \\ a_3 + a_4 &= b_{21} \\ a_4 &= b_{11} - b_{22}. \end{aligned}$$

To eliminate a_2 from the first equation, subtract equation two from equation one.

$$\begin{aligned} a_1 - \quad a_3 \quad\quad &= b_{11} - b_{12} \\ a_2 + a_3 + a_4 &= b_{12} \\ a_3 + a_4 &= b_{21} \\ a_4 &= b_{11} - b_{22}. \end{aligned}$$

To eliminate a_3 from equations one and two, add equation three to equation one and subtract equation three from equation two.

$$\begin{aligned} a_1 + \quad\quad a_4 &= b_{11} - b_{12} + b_{21} \\ a_2 \quad\quad &= b_{12} - b_{21} \\ a_3 + a_4 &= b_{21} \\ a_4 &= b_{11} - b_{22}. \end{aligned}$$

Finally, subtract equation four from equation one and equation three.

$$\begin{aligned} a_1 \quad\quad &= -b_{12} + b_{21} + b_{22} \\ a_2 \quad\quad &= b_{12} - b_{21} \\ a_3 \quad\quad &= b_{21} - b_{11} + b_{22} \\ a_4 &= b_{11} - b_{22}. \end{aligned}$$

Thus we have shown that given $B \in M_{2\times 2}(\mathbb{R})$, $B = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix}$, then

$$\begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} = (-b_{12} + b_{21} + b_{22}) \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + (b_{12} - b_{21}) \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$$

$$+(b_{21} - b_{11} + b_{22}) \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} + (b_{11} - b_{22}) \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}.$$

Hence $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$, $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, and $\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$ generate $M_{2\times 2}(\mathbb{R})$.

For example we can write $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$, where $b_{11} = 1$, $b_{12} = 2$, $b_{21} = 3$, $b_{22} = 4$ as

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = 5 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} + 6 \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} - 3 \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}.$$