The Gram-Schmidt Orthonormalization Process- HW Problems

In problems 1-3 apply the Gram-Schmidt process to the basis B to find an orthonormal basis for the inner product space V. Then write the vector v in terms of the orthonormal basis.

1. $V = \mathbb{R}^3$; $B = \{<1,1,0>, <0,1,1>, <1,0,1>\}$

   $v = <-2, 1, -1>$

2. $V = \mathbb{R}^3$; $B = \{<1,2,1>, <0,2,1>, <0,0,1>\}$

   $v = <2, 0, 1>$

3. $V = span(B) \subseteq \mathbb{R}^4$

   $B = \{<1,0,1,0>, <1,1,1,1>, <0,1,2,1>\}$

   $v = <-1,-1, 3, -1>$ (which is in the $span(B)$)

4. Use the Gram-Schmidt method to find an orthonormal basis for $V = P_2([0,1])$ from the basis $B = \{1, x, x^2\}$, with the inner product $<f,g> = \int_0^1 f(x)g(x)dx$.

5a. Show that $B = \{1, \cos(x), \sin(x)\} \subseteq C[-\pi, \pi]$ is a mutually orthogonal set with respect to the inner product
$< f, g >= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)g(x)dx$.

b. Turn this orthogonal set into an orthonormal set.

c. Find the Fourier coefficients of $f(x) = x$ with respect to this orthonormal set and approximate $f(x)$.

d. Find the Fourier coefficients of $f(x) = x^2$ with respect to this orthonormal set and approximate $f(x)$.

Note: when calculating Fourier coefficients with this inner product it can be useful to use the fact that $\int_{-\pi}^{\pi} h(x)dx = 0$ is $h(x)$ is an odd function (ie. $h(-x) = -h(x)$)

6. Let $\{v_1, \dots, v_n\}$ be an orthonormal basis for V. For any $v \in V$ show that $< v, v > = \sum_{i=1}^{n}(< v, v_i >)^2$.

This is sometimes called Parseval's identity.

7. Let $f, g \in C[-1,1]$ such that f is an even function (ie $f(-x) = f(x)$) and g is an odd function (ie $g(-x) = -g(x)$). Show that f is orthogonal to g with respect to the inner product

$$< f, g >= \int_{-1}^{1} f(x)g(x)dx.$$

8. Let $V = C[-1,1]$ with the inner product in problem 7.
Let $V' = P_2([-1,1]) \subseteq V = C[-1,1]$. Using the orthonormal basis $B = \{\frac{1}{\sqrt{2}}, \frac{\sqrt{6}}{2}x, \frac{\sqrt{10}}{4}(3x^2 - 1)\}$, find the function $g(x) \in V'$ which is the best approximation of $f(x) = \sin(\pi x) \in V$ (ie the $g(x)$ such that $\|\sin(\pi x) - g(x)\|$ is a minimum).