Changing Bases

Suppose V is a finite dimensional vector space with two different ordered bases $B_1 = \{w_1, \dots, w_n\}$ and $B_2 = \{v_1, \dots, v_n\}$. Let's call B_1 the old basis and B_2 the new basis. Given a vector $v \in V$ which is expressed as

$$v = b_1 v_1 + \cdots + b_n v_n$$

in the new basis, how do we express v in the old basis $v = a_1 w_1 + \cdots + a_n w_n$? That is, if we know $b_1, \dots, b_n$, how do we find $a_1, \dots, a_n$?

Clearly, if we can express each new basis vector v_i, $1 \leq i \leq n$, in terms of $w_1, \dots, w_n$, ie

$$v_i = c_{i1} w_1 + \cdots + c_{in} w_n$$

we can express v in terms of $w_1, \dots, w_n$.

Notice that we can think of changing bases from B_2 to B_1 as a linear transformation, $I: V \to V$, where I is just the identity map, $I(v) = v$, but we are using the basis $B_2 = \{v_1, \dots, v_n\}$ for the domain space and $B_1 = \{w_1, \dots, w_n\}$ as the basis for the range of I. It's also worth noting that when we represent I in matrix form, $[I]_{B_2}^{B_1}$, it does not look like $\begin{bmatrix} 1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 1 \end{bmatrix}$ (it only looks this way if $B_1 = B_2$).

Ex. Let $w_1 =< 1,2 >$, $w_2 =< 3,5 >$ be the old basis for $\mathbb{R}^2$ and $v_1 =< 1,-1 >$, $v_2 =< 1,-2 >$ be the new basis for $\mathbb{R}^2$. Express v_1 and v_2 in terms of w_1 and w_2.

$$v_1 =< 1,-1 >= c_{11} w_1 + c_{12} w_2$$

$$= c_{11} < 1,2 > + c_{12} < 3,5 >$$

$$< 1,-1 >=< c_{11} + c_{12},\ 2c_{11} + 5c_{12} >.$$

So we must solve a linear system of equations:

$$1 = \ \ c_{11} + 3c_{12}$$

$$-1 = 2c_{11} + 5c_{12}$$

$\Longrightarrow \ \ c_{11} = -8, \quad c_{12} = 3.$ So we have:

$$v_1 =< 1, -1 >= -8 < 1,2 > +3 < 3,5 >= -8w_1 + 3w_2.$$

$$v_2 =< 1, -2 >= c_{21}w_1 + c_{22}w_2$$

$$= c_{21} < 1,2 > +c_{22} < 3,5 >$$

$$< 1, -2 >=< c_{21} + 3c_{22},\ 2c_{21} + 5c_{22} >.$$

So we must solve a linear sytem of equations:

$$1 = \ \ c_{21} + 3c_{22}$$

$$-2 = 2c_{21} + 5c_{22}$$

$\Longrightarrow \ \ c_{21} = -11, \ \ c_{22} = 4.$ So we have:

$$v_2 =< 1, -2 >= -11 < 1,2 > +4 < 3,5 >= -11w_1 + 4w_2.$$

Notice that with respect to the (new) basis $B_2 = \{v_1, v_2\}$

$$v_1 =< 1,0 >_{B_2} = 1v_1 + 0v_2$$

$$v_2 =< 0,1 >_{B_2} = 0v_1 + 1v_2.$$

Thus if we let $P = \begin{bmatrix} -8 & -11 \\ 3 & 4 \end{bmatrix}$, where the i^{th} column of P is just the coordinates of v_i in the old basis w_1, w_2, then we have:

$$Pv_1 = \begin{bmatrix} -8 & -11 \\ 3 & 4 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -8 \\ 3 \end{bmatrix}$$

$$Pv_2 = \begin{bmatrix} -8 & -11 \\ 3 & 4 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} -11 \\ 4 \end{bmatrix}.$$

Thus P maps the coordinates of v_1 and v_2 in the basis $B_2 = \{v_1, v_2\}$ into coordinates of v_1, v_2 in the basis $B_1 = \{w_1 . w_2\}$.

Claim: P maps the coordinates of any vector $v \in V$ in the basis $B_2 = \{v_1, v_2\}$ into coordinates of v in the basis $B_1 = \{w_1 . w_2\}$.

Notice that if $v \in V$ and $v = b_1 v_1 + b_2 v_2$, $b_1, b_2 \in \mathbb{R}$, then since we know that

$$v_1 = -8w_1 + 3w_2$$

$$v_2 = -11w_1 + 4w_2$$

We have:

$$\begin{aligned} v &= b_1 v_1 + b_2 v_2 \\ &= b_1(-8w_1 + 3w_2) + b_2(-11w_1 + 4w_2) \\ &= (-8b_1 - 11b_2)w_1 + (3b_1 + 4b_2)w_2 \\ &= a_1 w_1 + a_2 w_2. \end{aligned}$$

That is:

$$Pv = \begin{bmatrix} -8 & -11 \\ 3 & 4 \end{bmatrix}\begin{bmatrix} b_1 \\ b_2 \end{bmatrix} = \begin{bmatrix} -8b_1 - 11b_2 \\ 3b_1 + 4b_2 \end{bmatrix} = \begin{bmatrix} a_1 \\ a_2 \end{bmatrix}.$$

So P maps the coordinates of any $v \in V$ in the basis $B_2 = \{v_1, v_2\}$ into coordinates of v in the basis $B_1 = \{w_1 . w_2\}$.

For example, if we have the vector $v = 4v_1 - 2v_2$ and we want to express this vector in terms of the basis $B_1 = \{w_1, w_2\}$ we get:

$$Pv = \begin{bmatrix} -8 & -11 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 4 \\ -2 \end{bmatrix} = \begin{bmatrix} -10 \\ 4 \end{bmatrix}.$$

Thus $v = -10w_1 + 4w_2$.

We can check that this is correct by expressing $v = 4v_1 - 2v_2 = -10w_1 + 4w_2$ in the standard basis for $\mathbb{R}^2$.

$$v = 4v_1 - 2v_2 = -4 < 1, -1 > -2 < 1, -2 >=< 2,0 >$$

$$v = -10w_1 + 4w_2 = -10 < 1,2 > +4 < 3,5 >=< 2,0 >.$$

So all we need to do is find each new basis vector v_i in terms of the old basis vectors $w_1, \dots, w_n$ and create the change of basis matrix P by letting the i^{th} column of P be coordinates of v_i in the old basis. So if

$$v_1 = a_{11}w_1 + a_{12}w_2 + \cdots + a_{1n}w_n$$

$$v_2 = a_{21}w_1 + a_{22}w_2 + \cdots + a_{2n}w_n$$

$$\vdots$$

$$v_n = a_{n1}w_1 + a_{n2}w_2 + \cdots + a_{nn}w_n.$$

Then the change of basis matrix P is given by:

$$P = \begin{bmatrix} a_{11} & a_{21} & \cdots & \cdots & a_{n1} \\ a_{12} & a_{22} & \cdots & \cdots & a_{n2} \\ a_{13} & a_{23} & a_{33} & \cdots & a_{n3} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{1n} & a_{2n} & a_{3n} & \cdots & a_{nn} \end{bmatrix}.$$

This is just the transpose of the coefficient matrix A,

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & \cdots & a_{2n} \\ a_{31} & a_{32} & a_{33} & \cdots & a_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & a_{n3} & \cdots & a_{nn} \end{bmatrix}$$

where $A\begin{bmatrix} w_1 \\ \vdots \\ w_n \end{bmatrix} = \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix}$.

In our example, P is a linear transformation of $\mathbb{R}^2$ onto $\mathbb{R}^2$ that maps the coordinates of vectors in the new basis $B_2 = \{v_1, v_2\}$ into their coordinates in the old basis $B_1 = \{w_1, w_2\}$. So if $v = b_1 v_1 + b_2 v_2 = a_1 w_1 + a_2 w_2$ then

$$P\begin{bmatrix} b_1 \\ b_2 \end{bmatrix} = \begin{bmatrix} a_1 \\ a_2 \end{bmatrix}.$$

Since $\{v_1, v_2\}$ and $\{w_1, w_2\}$ are bases for $\mathbb{R}^2$, P is invertible. Thus we also have

$$\begin{bmatrix} b_1 \\ b_2 \end{bmatrix} = P^{-1}\begin{bmatrix} a_1 \\ a_2 \end{bmatrix}.$$

That is, if we know the coordinates of a vector $v \in \mathbb{R}^2$ in the old basis, we can then find them in the new basis if we can find P^{-1} (this is also true in $\mathbb{R}^n$).

For a 2×2 matrix A where $\det(A) \neq 0$, it's easy to check that if

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

then

$$A^{-1} = \frac{1}{\det(A)}\begin{bmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{bmatrix}.$$

Ex. Using the bases $B_1 = \{w_1, w_2\} = \{< 1,2 >,\ < 3,5 >\}$ and $B_2 = \{v_1, v_2\} = \{< 1,-1 >, < 1,-2 >\}$ write the vector $v = 5w_1 - 2w_2$ in terms of v_1 and v_2. That is, find b_1 and b_2 such that $v = b_1 v_1 + b_2 v_2$.

In this example we are taking a vector in the old basis and writing it in terms of the new basis (earlier we went from the new basis to the old basis).

We know that the change of basis matrix from the new basis to the old basis:

$$P = \begin{bmatrix} -8 & -11 \\ 3 & 4 \end{bmatrix}.$$

To find the change of basis matrix from the old basis to the new basis we need P^{-1}:

$$P^{-1} = \frac{1}{\det(P)} \begin{bmatrix} 4 & 11 \\ -3 & -8 \end{bmatrix} = \begin{bmatrix} 4 & 11 \\ -3 & -8 \end{bmatrix}, \quad \text{since } \det(P) = 1.$$

$$P^{-1} \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} \ \Rightarrow\ \begin{bmatrix} 4 & 11 \\ -3 & -8 \end{bmatrix} \begin{bmatrix} 5 \\ -2 \end{bmatrix} = \begin{bmatrix} -2 \\ 1 \end{bmatrix}.$$

So $v = -2v_1 + v_2$.

Again we can check this by writing everything in the standard basis for $\mathbb{R}^2$:

$$v = 5w_1 - 2w_2 = 5 < 1,2 > -2 < 3,5 >$$
$$= < -1,0 >$$

$$v = -2v_1 + v_2 = -2 < 1,-1 > + < 1,-2 >$$
$$= < -1,0 >.$$

Now that we know how to write a vector in a different basis, how do we express a linear transformation in a new basis? That is, Let T be a linear transformation from V to V, where V is a finite dimensional vector space with an ordered basis $B_1 = \{w_1, \dots, w_n\}$. We know how to express T as an $n \times n$ matrix. How do we express T in a new basis $B_2 = \{v_1, \dots, v_n\}$?

Theorem: Let $T: V \to V$ be a linear transformation on a finite dimensional vector space V. Let B_1 and B_2 be ordered bases for V. Suppose P is the change of basis matrix that changes B_2 coordinates into B_1 coordinates, then

$$[T]_{B_2} = P^{-1}[T]_{B_1}P.$$

Proof:

$$\begin{aligned}
P[T]_{B_2} &= [I]_{B_2}^{B_1}[T]_{B_2}^{B_2} \\
&= [IT]_{B_2}^{B_1} \\
&= [TI]_{B_2}^{B_1} \\
&= [T]_{B_1}^{B_1}[I]_{B_2}^{B_1} \\
&= [T]_{B_1}P.
\end{aligned}$$

Thus we have: $$[T]_{B_2} = P^{-1}[T]_{B_1}P.$$

Ex. Let $\begin{bmatrix} 1 & 4 \\ 1 & 2 \end{bmatrix}$ represent a linear transformation $T$of $\mathbb{R}^2$ to $\mathbb{R}^2$ in the basis $w_1 =< 1,2 >$, $w_2 =< 3,5 >$. Find a matrix representation of T in the basis $v_1 =< 1,-1 >$, $v_2 =< 1,-2 >$.

We saw earlier that the change of basis matrix from from the new basis $B_2 = \{v_1, v_2\}$ to the old basis $B_1 = \{w_1, w_2\}$ was the matrix P

$$P = \begin{bmatrix} -8 & -11 \\ 3 & 4 \end{bmatrix}$$

and its inverse P^{-1}

$$P^{-1} = \begin{bmatrix} 4 & 11 \\ -3 & -8 \end{bmatrix}.$$

Thus in the basis v_1, v_2, T has the matrix representation:

$$\begin{aligned} P^{-1}[T]_{B_1}P &= \begin{bmatrix} 4 & 11 \\ -3 & -8 \end{bmatrix}\begin{bmatrix} 1 & 4 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} -8 & -11 \\ 3 & 4 \end{bmatrix} \\ &= \begin{bmatrix} 4 & 11 \\ -3 & -8 \end{bmatrix}\begin{bmatrix} 4 & 5 \\ -2 & -3 \end{bmatrix} \\ &= \begin{bmatrix} -6 & -13 \\ 4 & 9 \end{bmatrix}. \end{aligned}$$

We saw in an earlier example that with these two bases

$$v = 5w_1 - 2w_2 = -2v_1 + v_2.$$

Let's check that:

$$\begin{bmatrix} 1 & 4 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} 5 \\ -2 \end{bmatrix} = \begin{bmatrix} -3 \\ 1 \end{bmatrix} = -3w_1 + w_2$$

and $\begin{bmatrix} -6 & -13 \\ 4 & 9 \end{bmatrix}\begin{bmatrix} -2 \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \end{bmatrix} = -v_1 + v_2$

represent the same vector by writing them both in the standard basis for $\mathbb{R}^2$.

$$-3w_1 + w_2 = -3 < 1,2 > + < 3,5 > = < 0, -1 >$$

$$-v_1 + v_2 = - < 1, -1 > + < 1, -2 > = < 0, -1 > .$$

It's worh noting that we could have solved the previous problem without our change of basis formula, but it would have been messier. Let's see how.

We already solved for v_1 and v_2 in terms of w_1 and w_2 on pages 1 and 2:

$$v_1 = -8w_1 + 3w_2$$

$$v_2 = -11w_1 + 4w_2.$$

Since T is represented in the w_1, w_2 basis by $\begin{bmatrix} 1 & 4 \\ 1 & 2 \end{bmatrix}$ we can say:

$$T(v_1) = T(\ \ -8w_1 + 3w_2) = \begin{bmatrix} 1 & 4 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} -8 \\ 3 \end{bmatrix} = 4w_1 - 2w_2$$

$$T(v_2) = T(-11w_1 + 4w_2) = \begin{bmatrix} 1 & 4 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} -11 \\ 4 \end{bmatrix} = 5w_1 - 3w_2.$$

Now if we can represent $T(v_1), T(v_2)$ in terms of v_1, v_2 instead of w_1, w_2 we'll be just about done. So we need to solve for w_1, w_2 in terms of v_1, v_2.

We can do this by solving $w_1 = < 1,2 > = a < 1, -1 > + b < 1, -2 >$, etc. or by inverting the change of basis matrix $P = \begin{bmatrix} -8 & -11 \\ 3 & 4 \end{bmatrix}$, ie $P^{-1} = \begin{bmatrix} 4 & 11 \\ -3 & -8 \end{bmatrix}$.

Either way we get:

$$w_1 = \ \ 4v_1 - 3v_2$$

$$w_2 = 11v_1 - 8v_2.$$

Now we can plug into our formulas for $T(v_1), T(v_2)$.

$$T(v_1) = 4w_1 - 2w_2 = 4(4v_1 - 3v_2) - 2(11v_1 - 8v_2) = -6v_1 + 4v_2$$

$$T(v_2) = 5w_1 - 3w_2 = 5(4v_1 - 3v_2) - 3(11v_1 - 8v_2) = -13v_1 + 9v_2.$$

Thus in the v_1, v_2 basis we can represent T by $\begin{bmatrix} -6 & -13 \\ 4 & 9 \end{bmatrix}$,

as we found earlier.

Ex. Let $T: \mathbb{R}^2 \to \mathbb{R}^2$ by $T(< x_1, x_2 >) =< -x_1 + x_2,\ 2x_1 + 2x_2 >$.

a. Find the matrix representation of T with respect to the standard basis, B, for $\mathbb{R}^2$.

b. Let $v_1 =< 1, -1 >$ and $v_2 =< 1,2 >$. Find the matrix representation, E, of T with respect to the basis $B_1 = \{v_1, v_2\}$ (for both $\mathbb{R}^2{}'s$).

c. Let $w_1 =< 2,1 >$ and $w_2 =< 1,1 >$. Find the matrix representation, F, of T with respect to the basis $B_2 = \{w_1, w_2\}$ (for both $\mathbb{R}^2{}'s$).

d. Show using the matrices from parts b and c that you can find matrix F from matrix E and the change of basis matrix from $\{w_1, w_2\}$ to $\{v_1, v_2\}$.

a. $T(< 1,0 >) =< -1,2 >$

$T(< 0,1 >) =< 1,2 >$.

So: $A = [T]_B = \begin{bmatrix} -1 & 1 \\ 2 & 2 \end{bmatrix}$.

b. The change of basis matrix, P_1, from $B_1 = \{v_1, v_2\}$ to $B = \{e_1, e_2\}$ is:

$$v_1 =< 1, -1 >= e_1 - e_2$$

$$v_2 =< 1,2 > = e_1 + 2e_2$$

$$P_1 = \begin{bmatrix} 1 & 1 \\ -1 & 2 \end{bmatrix} \implies P_1^{-1} = \frac{1}{\det(P_1)} \begin{bmatrix} 2 & -1 \\ 1 & 1 \end{bmatrix} = \frac{1}{3} \begin{bmatrix} 2 & -1 \\ 1 & 1 \end{bmatrix}.$$

Using the change of basis formula we get a matrix representation of T in B_1:

$$E = P_1^{-1} A P_1 = \frac{1}{3} \begin{bmatrix} 2 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} -1 & 1 \\ 2 & 2 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ -1 & 2 \end{bmatrix}$$

$$= \frac{1}{3} \begin{bmatrix} 2 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} -2 & 1 \\ 0 & 6 \end{bmatrix} = \frac{1}{3} \begin{bmatrix} -4 & -4 \\ -2 & 7 \end{bmatrix}.$$

c. The change of basis matrix, P_2, from $B_2 = \{w_1, w_2\}$ to $B = \{e_1, e_2\}$ is:

$$w_1 =< 2,1 >= 2e_1 + e_2$$

$$w_2 =< 1,1 > = e_1 + e_2$$

$$P_2 = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} \implies P_2^{-1} = \frac{1}{\det(P_2)} \begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix}.$$

Using the change of basis formula we get a matrix representation of T in B_2:

$$F = P_2^{-1} A P_2 = \begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} -1 & 1 \\ 2 & 2 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} -1 & 0 \\ 6 & 4 \end{bmatrix} = \begin{bmatrix} -7 & -4 \\ 13 & 8 \end{bmatrix}.$$

d. Notice that P_2 is the change of basis matrix from $B_2 = \{w_1, w_2\}$ to $B = \{e_1, e_2\}$. Since P_1 is the change of basis matrix from $B_1 = \{v_1, v_2\}$ to $B = \{e_1, e_2\}$, P_1^{-1} is the change of basis matrix from $B = \{e_1, e_2\}$ to $B_1 = \{v_1, v_2\}$. Hence $P = P_1^{-1}P_2$ is the change of basis matrix from $B_2 = \{w_1, w_2\}$ to $B_1 = \{v_1, v_2\}$.

Thus we have:

$$P = P_1^{-1}P_2 = \frac{1}{3}\begin{bmatrix} 2 & -1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} = \frac{1}{3}\begin{bmatrix} 3 & 1 \\ 3 & 2 \end{bmatrix} \implies \quad P^{-1} = \begin{bmatrix} 2 & -1 \\ -3 & 3 \end{bmatrix}.$$

Hence we get:

$$\begin{aligned} P^{-1}EP &= \begin{bmatrix} 2 & -1 \\ -3 & 3 \end{bmatrix}\frac{1}{3}\begin{bmatrix} -4 & -4 \\ -2 & 7 \end{bmatrix}\frac{1}{3}\begin{bmatrix} 3 & 1 \\ 3 & 2 \end{bmatrix} \\ &= \frac{1}{9}\begin{bmatrix} 2 & -1 \\ -3 & 3 \end{bmatrix}\begin{bmatrix} -24 & -12 \\ 15 & 12 \end{bmatrix} \\ &= \frac{1}{3}\begin{bmatrix} 2 & -1 \\ -3 & 3 \end{bmatrix}\begin{bmatrix} -8 & -4 \\ 5 & 4 \end{bmatrix} \\ &= \frac{1}{3}\begin{bmatrix} -21 & -12 \\ 39 & 24 \end{bmatrix} = \begin{bmatrix} -7 & -4 \\ 13 & 8 \end{bmatrix} = F. \end{aligned}$$

Theorem: Let $T: V \to V$ be a linear transformation on a finite dimensional vector space V. Let $A = [T]_B$ in any ordered basis B. Then $\det(A)$ does not depend on the basis B.

Proof: For any change of basis we have

$$C = P^{-1}AP \quad \text{so we have:}$$

$$\begin{aligned} \det(C) &= \det(P^{-1}AP) \\ &= \det(P^{-1})\det(A)\det(P) \\ &= \frac{1}{\det(P)}\det(A)\det(P) \\ &= \det(A). \end{aligned}$$