Changing Bases- HW Problems

For all of the following problems when you are changing bases use a change of basis matrix P.

1. Let $T: \mathbb{R}^2 \to \mathbb{R}^2$ by $T(x_1, x_2) = (2x_1 + x_2,\ x_1 - x_2)$.

a. Find a matrix A which represent T in the standard basis $\{\vec{e}_1, \vec{e}_2\}$ for $\mathbb{R}^2$.

b. Let $v_1 = e_1 + e_2$ and $v_2 = e_1 - e_2$. $\{v_1, v_2\}$ is another basis for $\mathbb{R}^2$. Find a matrix representation, B, of T with respect to $\{v_1, v_2\}$ (for both $\mathbb{R}^2's$).

c. Let $w_1 = -e_1 + 3e_2$ and $w_2 = 2e_1 - e_2$. Find a matrix representation, C, of T with respect to $\{w_1, w_2\}$ (for both $\mathbb{R}^2's$).

d. Show that using the matrices from parts b and c you can find matrix C from matrix B and the change of basis matrix from $\{w_1, w_2\}$ to $\{v_1, v_2\}$.

In problems 2-5 A is a matrix representation of a linear transformation in the standard basis. Find a matrix representation of the linear transformation in the new basis.

2. $A = \begin{bmatrix} 2 & -1 \\ 1 & 3 \end{bmatrix}$; new basis= $\{<1,2>, <1,1>\}$

3. $A = \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix}$; new basis= $\{<7,3>, <2,1>\}$

4. $A = \begin{bmatrix} 1 & 0 & 2 \\ -1 & 1 & 3 \\ 0 & 1 & 1 \end{bmatrix}$;

new basis= $\{< 1,1,1 >, < 2,1,0 >, \ < -1,1,1 >\}$

You can assume that if $P = \begin{bmatrix} 1 & 2 & -1 \\ 1 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix}$ then $P^{-1} = \frac{1}{2}\begin{bmatrix} 1 & -2 & 3 \\ 0 & 2 & -2 \\ -1 & 2 & -1 \end{bmatrix}$

5. $A = \begin{bmatrix} 2 & 1 & -1 \\ 0 & 1 & 3 \\ -1 & 2 & 1 \end{bmatrix}$;

new basis= $\{< 0,-2,1 >, < 1,2,0 >, \ < 1,1,1 >\}$.

You can assume that if $P = \begin{bmatrix} 0 & 1 & 1 \\ -2 & 2 & 1 \\ 1 & 0 & 1 \end{bmatrix}$ then $P^{-1} = \begin{bmatrix} 2 & -1 & -1 \\ 3 & -1 & -2 \\ -2 & 1 & 2 \end{bmatrix}$

6. Suppose $T: P_2(\mathbb{R}) \to P_2(\mathbb{R})$ by

$$T(a_0 + a_1x + a_2x^2) = (a_0 + a_1) + (a_2 - a_1)x + (a_2 - a_0)x^2.$$

a. Find a matrix representation of T with respect to the standard basis for $P_2(\mathbb{R})$, $\{1,\ x,\ x^2\}$.

b. Find a matrix representation of T with respect to the basis $\{1 + x,\ x + x^2,\ 1 + x^2\}$.

You can assume that if $P = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}$ then $P^{-1} = \frac{1}{2}\begin{bmatrix} 1 & 1 & -1 \\ -1 & 1 & 1 \\ 1 & -1 & 1 \end{bmatrix}$.

c. Find a matrix representation of T with respect to the basis $\{1,\ 1 + x,\ 1 + x + x^2\}$.

You can assume that if $P = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$ then $P^{-1} = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}$.

7. $n \times n$ matrices A and B are similar if there exists an invertible $n \times n$ matrix Q such that $B = Q^{-1}AQ$. Show that if A and B are similar matrices then $\det(A) = \det(B)$.

8. Suppose that A, B, and C are $n \times n$ matrices. Show

a. A is similar to A

b. If A is similar to B, then B is similar to A.

c. If A is similar to B, and B is similar to C, then A is similar to C.

Note: a, b, and c mean that similarity of matrices is an equivalence relation.

9. Suppose that A is an invertible $n \times n$ matrix and A is similar to B.

a. Prove that B is invertible.

b. Prove that A^{-1} is similar to B^{-1}.