Determinants: An Overview

For each $n \times n$ matrix, A, we can calculate a number called the determinant of A, $det(A)$. This is often written as $|A|$.

Case 1. 1×1 matrices. If $A = [a]$ then $det(A) = |A| = a$.

Case 2. 2×2 matrices. Let $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$, the we have:

$$\det(A) = |A| = a_{11}a_{22} - a_{21}a_{12} .$$

Ex. $\begin{vmatrix} 3 & -2 \\ 4 & 5 \end{vmatrix} = 3(5) - (4)(-2) = 15 + 8 = 23.$

Def. Let $A = [a_{ij}]$ be an $n \times n$ matrix and let M_{ij} be the $(n-1) \times (n-1)$ matrix obtained from deleting the row and column containing $a_{ij.}$ The determinant of (M_{ij}) is called the **Minor** of a_{ij}. We define the **Cofactor**, A_{ij}, of a_{ij} by:

$$A_{ij} = (-1)^{i+j}\det(M_{ij}).$$

Def. The **Determinant of an $n \times n$ matrix A**, where $A = \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nn} \end{bmatrix}$, is given by:

$$Det(A) = a_{11}A_{11} + a_{12}A_{12} + \cdots + a_{1n}A_{1n}$$

where $A_{1j} = (-1)^{1+j}\det(M_{1j})$; $\quad j = 1,2,\cdots,n.$

Case 3. 3x3 matrices. Let A be:

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

Then $det(A) = a_{11}A_{11} + a_{12}A_{12} + a_{13}A_{13}$;

where: $A_{11} = (-1)^{1+1} \det \begin{bmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{bmatrix} = \det \begin{bmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{bmatrix}$

$$A_{12} = (-1)^{1+2} \det \begin{bmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{bmatrix} = -\det \begin{bmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{bmatrix}$$

$$A_{13} = (-1)^{1+3} \det \begin{bmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{bmatrix} = \det \begin{bmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{bmatrix}.$$

Ex. Find $det(A)$ where $A = \begin{bmatrix} 2 & 1 & 3 \\ 4 & 1 & 2 \\ 1 & 2 & 3 \end{bmatrix}$.

$$\begin{vmatrix} 2 & 1 & 3 \\ 4 & 1 & 2 \\ 1 & 2 & 3 \end{vmatrix} = 2\begin{vmatrix} 1 & 2 \\ 2 & 3 \end{vmatrix} - 1\begin{vmatrix} 4 & 2 \\ 1 & 3 \end{vmatrix} + 3\begin{vmatrix} 4 & 1 \\ 1 & 2 \end{vmatrix}$$

$$= 2[(1)(3) - 2(2)] - 1[4(3) - 1(2)] + 3[4(2) - 1(1)]$$

$$= 2(3 - 4) - 1(12 - 2) + 3(8 - 1)$$

$$= 2(-1) - 10 + 3(7)$$

$$= 9.$$

Ex. Find the determinant of $A = \begin{bmatrix} 1 & 0 & 0 & 3 \\ 2 & 0 & 1 & -2 \\ -1 & 1 & 0 & 0 \\ 0 & 3 & 1 & 3 \end{bmatrix}$.

$$|A| = 1\begin{vmatrix} 0 & 1 & -2 \\ 1 & 0 & 0 \\ 3 & 1 & 3 \end{vmatrix} - 0\begin{vmatrix} 2 & 1 & -2 \\ -1 & 0 & 0 \\ 0 & 1 & 3 \end{vmatrix} + 0\begin{vmatrix} 2 & 0 & -2 \\ -1 & 1 & 0 \\ 0 & 3 & 3 \end{vmatrix} - 3\begin{vmatrix} 2 & 0 & 1 \\ -1 & 1 & 0 \\ 0 & 3 & 1 \end{vmatrix}$$

$$= 1\left[0\begin{vmatrix} 0 & 0 \\ 1 & 3 \end{vmatrix} - 1\begin{vmatrix} 1 & 0 \\ 3 & 3 \end{vmatrix} - 2\begin{vmatrix} 1 & 0 \\ 3 & 1 \end{vmatrix}\right]$$

$$-3\left[2\begin{vmatrix} 1 & 0 \\ 3 & 1 \end{vmatrix} - 0\begin{vmatrix} -1 & 0 \\ 0 & 1 \end{vmatrix} + 1\begin{vmatrix} -1 & 1 \\ 0 & 3 \end{vmatrix}\right]$$

$$= 1[0 - 1(3 - 0) - 2(1 - 0)] - 3[2(1 - 0) - 0 + 1(-3 - 0)]$$

$$= 1[-3 - 2] - 3[2 - 3] = -5 + 3 = -2.$$

Theorem: If A is an $n \times n$ matrix with $n \geq 2$ then $det(A)$ can be expressed as a cofactor expansion using any column or row.

$Det(A) = a_{i1}A_{i1} + a_{i2}A_{i2} + \cdots + a_{in}A_{in}$ (using the ith row); or

$Det(A) = a_{1j}A_{1j} + a_{2j}A_{2j} + \cdots + a_{nj}A_{nj}$ (using the jth column).

Ex Find $det(A)$ where $A = \begin{bmatrix} 2 & 1 & 3 \\ 4 & 1 & 2 \\ 1 & 2 & 3 \end{bmatrix}$

a. Using the 3rd row
b. Using the 2nd column

a. $$det(A\,) = (-1)^{3+1}\begin{vmatrix} 1 & 3 \\ 1 & 2 \end{vmatrix} - 2\begin{vmatrix} 2 & 3 \\ 4 & 2 \end{vmatrix} + 3\begin{vmatrix} 2 & 1 \\ 4 & 1 \end{vmatrix}$$
$$= 1\,(2-3) \;-\; 2(4-12) \;+\; 3(2-4)$$
$$= 1\,(-1) \;-2(-8) \;+3(-2) = -1 \;+16 \;-6 = 9$$

b. $$det(A) \;= (-1)^{1+2}\begin{vmatrix} 4 & 2 \\ 1 & 3 \end{vmatrix} + 1\begin{vmatrix} 2 & 3 \\ 1 & 3 \end{vmatrix} - 2\begin{vmatrix} 2 & 3 \\ 4 & 2 \end{vmatrix}$$
$$= -1\,(12-2) \;+1(6-3) - 2(4-12)$$
$$= \;-1\,(10) + 3 - 2(-8) = \;-10 + 3 + 16 = 9.$$

Ex. Find $det(A)$ where $A = \begin{bmatrix} 1 & 0 & 2 & 3 \\ 0 & 0 & 1 & -4 \\ 2 & 0 & 0 & 1 \\ -2 & 3 & 0 & 1 \end{bmatrix}$ using the 2nd column.

$$detA = (-1)^{1+2}(0)\begin{vmatrix} 0 & 1 & -4 \\ 2 & 0 & 1 \\ -2 & 0 & 1 \end{vmatrix} + (-1)^{2+2}(0)\begin{vmatrix} 1 & 2 & 3 \\ 2 & 0 & 1 \\ -2 & 0 & 1 \end{vmatrix}$$
$$+(-1)^{3+2}(0)\begin{vmatrix} 1 & 2 & 3 \\ 0 & 1 & -4 \\ -2 & 0 & 1 \end{vmatrix} + (-1)^{4+2}(3)\begin{vmatrix} 1 & 2 & 3 \\ 0 & 1 & -4 \\ 2 & 0 & 1 \end{vmatrix}$$

$$= 3\left[1(1-0) - 2\big(-0(-8)\big) + 3(0-2)\right]$$

$$= \;3\,[1 - 2(8) + 3(-2)] = 2(1 - 16 - 6) = \;3(-21) = \;-63.$$

Theorem If A is an $n \times n$ matrix then $det(A^t) = det(A)$

Ex. $$det\begin{bmatrix}1 & 2 & 3\\4 & 5 & 6\\7 & 8 & 9\end{bmatrix} = det\begin{bmatrix}1 & 4 & 7\\2 & 5 & 8\\3 & 6 & 9\end{bmatrix}.$$

Theorem: If A is an $n \times n$ matrix which is either upper triangular or lower triangular then $det(A) =$ product of the diagonal elements.

Proof:

$$det\begin{bmatrix}a_{11} & a_{12} & \cdots & \cdots & a_{1n}\\0 & a_{22} & \cdots & \cdots & a_{2n}\\0 & 0 & a_{33} & \cdots & a_{3n}\\0 & 0 & 0 & \ddots & \vdots\\0 & 0 & 0 & \cdots & a_{nn}\end{bmatrix}$$ taking cofactors along the first column

$$= a_{11}\begin{vmatrix}a_{22} & \cdots & \cdots & a_{2n}\\0 & a_{33} & \cdots & a_{3n}\\\vdots & 0 & \ddots & \vdots\\0 & 0 & \cdots & a_{nn}\end{vmatrix} = a_{11}a_{22}\begin{vmatrix}a_{33} & \cdots & \cdots & a_{3n}\\0 & \ddots & & \vdots\\\vdots & 0 & \ddots & \\0 & 0 & \cdots & a_{nn}\end{vmatrix}$$

$$= a_{11}a_{22}\ldots a_{nn}.$$

Ex. $$det\begin{bmatrix}1 & 5 & 7\\0 & 2 & 6\\0 & 0 & 3\end{bmatrix} = 1(2)(3) = 6$$

Ex. $$det\begin{bmatrix}2 & 0 & 0\\6 & -1 & 0\\2 & 5 & 4\end{bmatrix} = 2(-1)(4) = -8.$$

Theorem Let A be an $n \times n$ matrix

i. If A has an entire row or column of zeros then $det(A) = 0$
ii. If A has 2 identical rows or columns then $det(A) = 0$.

Ex. $det \begin{bmatrix} 1 & 4 & 2 & 0 \\ 9 & 3 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 2 & 1 & -3 & 2 \end{bmatrix} = 0$

$det \begin{bmatrix} 1 & 0 & 1 & -1 \\ 1 & 4 & 1 & 2 \\ 3 & 2 & 3 & -3 \\ 2 & 9 & 2 & 4 \end{bmatrix} = 0.$

Theorem: If A and B are $n \times n$ matrices then $\det(AB) = (\det(A))(\det(B))$.

Theorem An $n \times n$ matrix A is invertible (nonsingular) if and only if $det(A) \neq 0$. If A is invertible then $\det(A^{-1}) = \frac{1}{\det(A)}$.

Ex. Determine if A is invertible where

$$A = \begin{bmatrix} 1 & 2 & 1 \\ -2 & 1 & 1 \\ 3 & 2 & 1 \end{bmatrix}.$$

$$\begin{aligned} det(A) &= 1\begin{bmatrix} 1 & 1 \\ 2 & 1 \end{bmatrix} - 2\begin{bmatrix} -2 & 1 \\ 3 & 1 \end{bmatrix} + 1\begin{bmatrix} -2 & 1 \\ 3 & 2 \end{bmatrix} \\ &= 1(1-2) - 2(-2-3) + 1(-4-3) \\ &= -1 - 2(-5) + (-7) = -1 + 10 - 7 = 2 \neq 0. \end{aligned}$$

So A is invertible.

Notice that an $n \times n$ matrix A is invertible precisely when the $Rank(A) = n$. Thus the n column vectors of A are linearly independent if and only if A is invertible. Hence the n column vectors of A are linearly independent if and only if $det(A) \neq 0$. So given any n vectors, $v_1, \dots, v_n$ in $\mathbb{R}^n$ we can test to see if they are linearly independent by calculating the determinant of the matrix formed by making $v_1, \dots, v_n$ the columns of matrix. If the determinant is nonzero the vectors are linearly independent. If the determinant is zero the vectors are dependent.

Ex. Determine if the vectors $v_1 =< 1,0,-2 >$, $v_2 =< 0,3,1 >$, and $v_3 =< 2 - 1,3 >$ are linearly independent.

Let $A = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 3 & -1 \\ -2 & 1 & 3 \end{bmatrix}$ where the columns of A are v_1, v_2, v_3.

$$det(A) = (1)\begin{vmatrix} 3 & -1 \\ 1 & 3 \end{vmatrix} - (0)\begin{vmatrix} 0 & -1 \\ -2 & 3 \end{vmatrix} + 2\begin{vmatrix} 0 & 3 \\ -2 & 1 \end{vmatrix}$$

$$= (1)(9 + 1) + 0 + 2(0 + 6)$$

$$= 22 \neq 0.$$

Thus v_1, v_2 and v_3 are linearly independent.

Theorem: If B is a matrix obtained by interchanging any two rows or columns of an $n \times n$ matrix A then $\det(B) = -\det(A)$. In particular if two rows or columns of A are identical then $\det(A) = 0$.

Ex. $$det\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} = -det\begin{bmatrix} 7 & 8 & 9 \\ 4 & 5 & 6 \\ 1 & 2 & 3 \end{bmatrix}$$

Theorem: If B is a matrix obtained by multiplying each entry of a row or column of an $n \times n$ matrix A by $k \in \mathbb{R}$, then $\det(B) = k\det(A)$. In particular, if $B = kA$ then $\det(B) = k^n \det(A)$.

Ex. $$det\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 21 & 24 & 27 \end{bmatrix} = 3det\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}; \quad det\begin{bmatrix} 3 & 6 \\ 9 & 12 \end{bmatrix} = 3^2 det\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}.$$

Theorem: If B is a matrix obtained from an $n \times n$ matrix A by adding a multiple of row i to row j or adding a multiple of column i to column j, $i \neq j$, Then $\det(B) = \det(A)$.

Thus one can use elementary row operations to put a matrix in upper (or lower) triangular form to calculate the determinant of an $n \times n$ matrix.

Ex. Evaluate $det\begin{bmatrix} 1 & -1 & 2 & 1 \\ 2 & -1 & -1 & 4 \\ -4 & 5 & -10 & -6 \\ 3 & -2 & 10 & -1 \end{bmatrix}$ using elementary row operations.

$$\begin{bmatrix} 1 & -1 & 2 & 1 \\ 2 & -1 & -1 & 4 \\ -4 & 5 & -10 & -6 \\ 3 & -2 & 10 & -1 \end{bmatrix} \xrightarrow[R_2-2R_1\to R_2]{} \begin{bmatrix} 1 & -1 & 2 & 1 \\ 0 & 1 & -5 & 2 \\ -4 & 5 & -10 & -6 \\ 3 & -2 & 10 & -1 \end{bmatrix}$$

$$\xrightarrow[R_3+4R_1\to R_3]{} \begin{bmatrix} 1 & -1 & 2 & 1 \\ 0 & 1 & -5 & 2 \\ 0 & 1 & -2 & -2 \\ 3 & -2 & 10 & -1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -1 & 2 & 1 \\ 0 & 1 & -5 & 2 \\ 0 & 1 & -2 & -2 \\ 3 & -2 & 10 & -1 \end{bmatrix} \xrightarrow[R_4-3R_1\to R_4]{} \begin{bmatrix} 1 & -1 & 2 & 1 \\ 0 & 1 & -5 & 2 \\ 0 & 1 & -2 & -2 \\ 0 & 1 & 4 & -4 \end{bmatrix}$$

$$\xrightarrow[R_4-R_3\to R_4]{} \begin{bmatrix} 1 & -1 & 2 & 1 \\ 0 & 1 & -5 & 2 \\ 0 & 1 & -2 & -2 \\ 0 & 0 & 6 & -2 \end{bmatrix}$$

$$\xrightarrow[R_3-R_2\to R_3]{} \begin{bmatrix} 1 & -1 & 2 & 1 \\ 0 & 1 & -5 & 2 \\ 0 & 0 & 3 & -4 \\ 0 & 0 & 6 & -2 \end{bmatrix}$$

$$\xrightarrow[R_4-2R_3\to R_4]{} \begin{bmatrix} 1 & -1 & 2 & 1 \\ 0 & 1 & -5 & 2 \\ 0 & 0 & 3 & -4 \\ 0 & 0 & 0 & 6 \end{bmatrix}.$$

Notice that every elementary row operation we used was of type 3, thus the value of the determinant didn't change through this process. Hence:

$$det\begin{bmatrix} 1 & -1 & 2 & 1 \\ 2 & -1 & -1 & 4 \\ -4 & 5 & -10 & -6 \\ 3 & -2 & 10 & -1 \end{bmatrix} = det\begin{bmatrix} 1 & -1 & 2 & 1 \\ 0 & 1 & -5 & 2 \\ 0 & 0 & 3 & -4 \\ 0 & 0 & 0 & 6 \end{bmatrix}$$

$$= (1)(1)(3)(6) = 18.$$