# Gaussian Elimination and Row Echelon Form

In this section we describe a method of solving systems of linear equations called **Gaussian elimination**. We start by writing a system of linear equations in augmented form.

Ex. Write the following system of linear equations in augmented form:

$$\begin{aligned} 3x_1 - 12x_2 \qquad\quad - 3x_4 &= 12 \\ x_1 - \ 4x_2 - \ x_3 + \quad x_4 &= \ 3 \\ 2x_1 - \ 8x_2 + 4x_3 - 10x_4 &= 12. \end{aligned}$$

If the system is given by $Ax = b$ then the augmented matrix for this system is:

$$[A|b] = \left[\begin{array}{cccc|c} 3 & -12 & 0 & -3 & 12 \\ 1 & -4 & -1 & 1 & 3 \\ 2 & -8 & 4 & -10 & 12 \end{array}\right].$$

Def. A matrix is said to be in **Row Echelon Form** if:

1. If the first nonzero entry in each nonzero row is 1
2. If row $k$ does not consist of entirely of zeros, the number of leading zero entries in row $k + 1$ is greater than the number of leading zeros entries in row $k$.
3. If there are rows whose entries are all zero, they are below the rows having nonzero entries.

Notice that 2 and 3 mean the number of zeros preceding the first nonzero entry (which is a 1 by number 1) in each row increases row by row until only zero rows remain.

Examples of matrices in Row Echelon Form:

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{bmatrix} \quad \begin{bmatrix} 1 & -1 & 2 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \quad \begin{bmatrix} 1 & -1 & 0 & 2 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Examples of matrices NOT in Row Echelon Form:

$$\begin{bmatrix} 2 & 2 & 3 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{bmatrix} \quad \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 2 \end{bmatrix} \quad \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

Def. A matrix is in **Reduced Row Echelon Form** if:

1. The matrix is in Row Echelon Form
2. The first nonzero entry in each row is the only nonzero entry in its column

Examples of matrices in reduced row echelon form:

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \begin{bmatrix} 1 & 0 & 0 & 4 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 0 \end{bmatrix} \quad \begin{bmatrix} 1 & 2 & 0 & 4 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Gaussian elimination is a process where we use elementary row operations to transform the original augmented matrix into row echelon form and then continue to transform it into reduced row echelon form.

We begin this process by creating a 1 in the left most nonzero column and then zeros underneath it.

Ex.
$$\left[\begin{array}{cccc|c} 3 & -12 & 0 & -3 & 12 \\ 1 & -4 & -1 & 1 & 3 \\ 2 & -8 & 4 & -10 & 12 \end{array}\right] \xrightarrow[R_1 \leftrightarrow R_2]{} \left[\begin{array}{cccc|c} 1 & -4 & -1 & 1 & 3 \\ 3 & -12 & 0 & -3 & 12 \\ 2 & -8 & 4 & -10 & 12 \end{array}\right]$$
$$\xrightarrow[R_2 - 3R_1 \to R_2]{} \left[\begin{array}{cccc|c} 1 & -4 & -1 & 1 & 3 \\ 0 & 0 & 3 & -6 & 3 \\ 2 & -8 & 4 & -10 & 12 \end{array}\right]$$
$$\xrightarrow[R_3 - 2R_1 \to R_3]{} \left[\begin{array}{cccc|c} 1 & -4 & -1 & 1 & 3 \\ 0 & 0 & 3 & -6 & 3 \\ 0 & 0 & 6 & -12 & 6 \end{array}\right].$$

Now create a 1 in the next row in the left most possible column, without using rows above it.

$$\left[\begin{array}{cccc|c} 1 & -4 & -1 & 1 & 3 \\ 0 & 0 & 3 & -6 & 3 \\ 0 & 0 & 6 & -12 & 6 \end{array}\right] \xrightarrow[\frac{1}{3}R_2 \to R_2]{} \left[\begin{array}{cccc|c} 1 & -4 & -1 & 1 & 3 \\ 0 & 0 & 1 & -2 & 1 \\ 0 & 0 & 6 & -12 & 6 \end{array}\right].$$

Next create zeros in the column below the 1 in the previous step.

$$\left[\begin{array}{cccc|c} 1 & -4 & -1 & 1 & 3 \\ 0 & 0 & 1 & -2 & 1 \\ 0 & 0 & 6 & -12 & 6 \end{array}\right] \xrightarrow[R_3 - 6R_2 \to R_3]{} \left[\begin{array}{cccc|c} 1 & -4 & -1 & 1 & 3 \\ 0 & 0 & 1 & -2 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right].$$

Repeat these steps to create 1's with zeros in the column below for all successive row that have any nonzero entries.

This matrix is now in row echelon form.

To put this matrix in reduced row echelon form we need to create zeros above the first nonzero entry of each row.

$$\left[\begin{array}{cccc|c} 1 & -4 & -1 & 1 & 3 \\ 0 & 0 & 1 & -2 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \xrightarrow[R_1+R_2 \to R_1]{} \left[\begin{array}{cccc|c} 1 & -4 & 0 & 1 & 4 \\ 0 & 0 & 1 & -2 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

The augmented matrix is now in reduced row echelon form. This gives us the following equations:

$$x_1 - 4x_2 \qquad -x_4 = 4$$

$$x_3 - 2x_4 = 1.$$

Or

$$x_1 = 4 + 4x_2 + x_4$$

$$x_2 = x_2$$

$$x_3 = 1 + 2x_4$$

$$x_4 = x_4.$$

If we let $x_2 = a$ and $x_4 = b$ (since $x_2$ and $x_4$ can be any real number), we have:

All Solutions$=< 4 + 4a + b,\ a,\ 1 + 2b,\ b >$

$$=< 4, 0, 1, 0 > +a < 4, 1, 0, 0 > +b < 1, 0, 2, 1 >;\ a, b \in \mathbb{R}.$$

Notice that $< 4,0,1,0 >$ is a specific solution to the system of equations and $< 4, 1, 0, 0 >$ and $< 1, 0, 2, 1 >$ span the nullspace of the associated homogeneous system.

Ex. Solve the following system of equations using Gaussian elimination:

$$x_1 + x_2 + x_3 + \ \ x_4 + \ \ x_5 = 2$$

$$x_1 + x_2 + x_3 + 2x_4 + 2x_5 = 3$$

$$x_1 + x_2 + x_3 + 2x_4 + 3x_5 = 2.$$

Augmented Matrix= $\left[\begin{array}{ccccc|c} 1 & 1 & 1 & 1 & 1 & 2 \\ 1 & 1 & 1 & 2 & 2 & 3 \\ 1 & 1 & 1 & 2 & 3 & 2 \end{array}\right]$.

Let's put this matrix in row echelon form.

$$\left[\begin{array}{ccccc|c} 1 & 1 & 1 & 1 & 1 & 2 \\ 1 & 1 & 1 & 2 & 2 & 3 \\ 1 & 1 & 1 & 2 & 3 & 2 \end{array}\right] \xrightarrow[R_2 - R_1 \to R_2]{} \left[\begin{array}{ccccc|c} 1 & 1 & 1 & 1 & 1 & 2 \\ 0 & 0 & 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 2 & 3 & 2 \end{array}\right]$$

$$\xrightarrow[R_3 - R_1 \to R_3]{} \left[\begin{array}{ccccc|c} 1 & 1 & 1 & 1 & 1 & 2 \\ 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 2 & 0 \end{array}\right]$$

$$\xrightarrow[R_3 - R_2 \to R_3]{} \left[\begin{array}{ccccc|c} 1 & 1 & 1 & 1 & 1 & 2 \\ 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & -1 \end{array}\right].$$

The last matrix is in row echelon form.

Now let's put the augmented matrix into reduced row echelon form.

$$\left[\begin{array}{ccccc|c} 1 & 1 & 1 & 1 & 1 & 2 \\ 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & -1 \end{array}\right] \xrightarrow[R_1 - R_2 \to R_1]{} \left[\begin{array}{ccccc|c} 1 & 1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & -1 \end{array}\right]$$

$$\xrightarrow[R_2 - R_3 \to R_2]{} \left[\begin{array}{ccccc|c} 1 & 1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 0 & 1 & -1 \end{array}\right].$$

Now the matrix is in reduced row echelon form with **free variables** $x_2, x_3$, that is, there are no constraints on $x_2$ or $x_3$.

Write down the equations from this matrix and solve all lead variables in terms of $x_2, x_3$ and constants.

$x_1 + x_2 + x_3 = 1$; So we have: $x_1 = 1 - x_2 - x_3$

$x_4 = 2$

$x_5 = -1.$

Let's say $x_2$ can be any real number $a$, and $x_3$ can be any real number $b$, then all of the solutions are of the form:

$< 1 - a - b,\ a,\ b,\ 2,\ -1 >$; where $a, b$ are any real numbers.

We can also write the solution set as:

$$< 1 - a - b, a, b, 2, -1 >= \\ < 1,0,0,2,-1 > +a < -1,1,0,0,0 > +b < -1,0,1,0,0 >.$$

where $< 1,0,0,2,-1 >$ is a specific solution to the system and the vectors $< -1,1,0,0,0 >$ and $< -1,0,1,0,0 >$ span the null space of the associated homogenous system.

In the case of an inconsistent system of equations (ie, there is no solution), the Gaussian elimination process will lead to a row with all zeros and a nonzero element in the far right slot (to the right of the vertical line in the augmented matrix).