Solving Systems of Linear Equations Using Linear Transformations

A linear system of m equations in n unknowns is given by:

$$a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1$$
$$a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2$$
$$\vdots$$
$$a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = b_m$$

where a_{ij} and $b_i,\ 1 \le i \le m,\ 1 \le j \le n$ are real numbers and the unknowns, $x_1, \dots, x_n$ are also real numbers.

Notice that we can write these equations in matrix form as $Ax = b$, where A is the $m \times n$ matrix, x is a vector in $\mathbb{R}^n$ and b is a vector in $\mathbb{R}^m$ given by:

$$A = \begin{bmatrix} a_{11} & a_{12} \cdots & a_{1n} \\ a_{21} & a_{22} & a_{2n} \\ \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} \cdots & a_{mn} \end{bmatrix}; \quad x = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}; \quad b = \begin{bmatrix} b_1 \\ \vdots \\ b_m \end{bmatrix}.$$

So a solution to the system of linear equations given by $Ax = b$ is any n-tuple

$$s = \begin{bmatrix} s_1 \\ \vdots \\ s_n \end{bmatrix} \in \mathbb{R}^n \text{ such that } As = b.$$

The system is called **consistent** if the solution set is non-empty, otherwise it is called **inconsistent**.

Ex. Consider the system of equations:

$$x_1 + x_2 = 4$$

$$x_1 - x_2 = 2.$$

It is clear from elementary algebra that $x_1 = 3$ and $x_2 = 1$ is the solution set.

In matrix form we have:

$$\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 4 \\ 2 \end{bmatrix}$$

where $A = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$, $b = \begin{bmatrix} 4 \\ 2 \end{bmatrix}$, and $s = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$.

Ex. Consider the system of equations:

$$2x_1 - x_2 + x_3 = 6$$

$$x_1 + x_2 + x_3 = 9.$$

In matrix form we write:

$$\begin{bmatrix} 2 & -1 & 1 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 6 \\ 9 \end{bmatrix}.$$

This system has an infinite number of solutions. Among them are:

$$s = \begin{bmatrix} 5 \\ 4 \\ 0 \end{bmatrix} \quad \text{and} \quad s = \begin{bmatrix} 7 \\ 5 \\ -3 \end{bmatrix}.$$

Later we will see how to express all of the solutions.

Ex. Consider the system of equations:

$$x_1 + x_2 = 1$$

$$x_1 + x_2 = 2$$

which in matrix form is:

$$\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}.$$

This system of equations does not have any solution since if the sum of two numbers is 1 it can't also be 2. Thus this system is inconsistent.

Def. A system $Ax = b$ of m linear equations in n unknowns is said to be **homogeneous** if $b = 0$. Otherwise the system is said to be **nonhomogeneous**.

A homogenous system has at least one solution, $x_1 = x_2 = \cdots = x_n = 0$.

Given a system of m linear equations in n unknowns, $Ax = b$, we can think of A as representing a linear transformation $L_A: \mathbb{R}^n \to \mathbb{R}^m$. Thus solving the system is equivalent to finding all vectors in $\mathbb{R}^n$ whose image under L_A is the fixed vector $< b_1, \dots, b_m >$. In particular, if $< b_1, \dots, b_m > = < 0, \dots, 0 >$ then the solution is the null set, $N(L_A)$, of L_A. Thus we have:

Theorem: Let $Ax = 0$ be a homogeneous system of m linear equations in n unknowns. Let K denote the set of all solutions to $Ax = 0$. Then $K = N(L_A)$, hence K is a subspace of $\mathbb{R}^n$ of dimension $n - Rank(L_A) = n - Rank(A)$.

Corollary: If $m < n$, the system $Ax = 0$ has a nonzero solution.

Proof: $\dim(K) = n - Rank(L_A) \geq n - m > 0$.

Thus a homogenous system of linear equations with fewer equations than unknowns always has a nonzero solution.

Ex. Consider the system:

$$2x_1 - x_2 + x_3 = 0$$
$$x_1 + x_2 + x_3 = 0.$$

Find all of the solutions.

In matrix form we have:

$$Ax = \begin{bmatrix} 2 & -1 & 1 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

$< 2,1 >$ and $< -1,1 >$ (the first two column vectors) are linearly independent and $\dim(R(L_A)) \leq 2$ so $\dim\big(R(L_A)\big) = 2$.

Thus the dimesion of the solution set $K = N(L_A)$ is:

$$\dim(K) = n - Rank(A) = 3 - 2 = 1.$$

So any nonzero vector in K will be a basis for K. So we just need a nonzero vector in $N(L_A)$.

To find a nonzero vector in $N(L_A)$ we can just let $x_1 = 1$, for example, in the system to get:

$$2 - x_2 + x_3 = 0 \quad \text{or} \quad -x_2 + x_3 = -2$$
$$1 + x_2 + x_3 = 0 \qquad\qquad x_2 + x_3 = -1.$$

Solving these equations we get: $x_2 = \frac{1}{2}$, $x_3 = -\frac{3}{2}$.

Thus $< 1, \frac{1}{2}, -\frac{3}{2} >$ is a nonzero vector in $N(L_A)$.

Thus the solution set for this system is: $K = \left\{ t < 1, \frac{1}{2}, -\frac{3}{2} > \middle|\ t \in \mathbb{R} \right\}$.

Ex. Consider the system $2x_1 - x_2 + x_3 = 0$ of one equation in three unknowns.

In matrix form we have: $\begin{bmatrix} 2 & -1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = [0]$. Find all of the solutions.

$Rank(A) = 1$ so $\dim(K) = n - Rank(A) = 3 - 1 = 2$.

Thus if we can find two linearly independent solutions they will span K.

We can find one solution by letting $x_1 = 0$. Then

$$-x_2 + x_3 = 0 \quad \Longrightarrow \quad x_2 = x_3 \quad \Longrightarrow < 0,1,1 > \text{ is a solution.}$$

Now let $x_3 = 0$. Then $2x_1 - x_2 = 0 \quad \Longrightarrow \quad 2x_1 = x_2$.

Thus $< 1,2,0 >$ is another solution.

Since $< 0,1,1 >$ and $< 1,2,0 >$ are linearly independent we know the solution set is:

$$K = \{t_1 < 0,1,1 > +t_2 < 1,2,0 >| \; t_1, t_2 \in \mathbb{R}\}.$$

Theorem: Let K be the solution set for the system of linear equations $Ax = b$. Let K_H be the solution set for $Ax = 0$. Then if s is any solution to $Ax = b$ then all solutions, K, are given by:

$$K = \{s\} + K_H = \{s + k| \; k \in K_H\}.$$

Proof: Let s and w be any two solutions to $Ax = b$.

Then we have:

$$A(s - w) = A(s) - A(w) = b - b = 0.$$

So $s - w \in K_H \quad \Longrightarrow s = w + k$ for some $k \in K_H$. $\quad \Longrightarrow K \subseteq \{s\} + K_H$.

Now suppose that $w \in \{s\} + K_H$.

Then $w = s + k$, for some $k \in K_H$. So we have:

$$Aw = A(s+k) = As + Ak = b, \quad \Longrightarrow \quad w \in K.$$

Hence $\{s\} + K_H \subseteq K, \quad \Longrightarrow \quad \{s\} + K_H = K.$

Ex. Consider the system

$$2x_1 - x_2 + x_3 = 6$$

$$x_1 + x_2 + x_3 = 9.$$

Find all of the solutions.

We saw earlier that $K_H = \left\{ t < 1, \frac{1}{2}, -\frac{3}{2} > \middle| \ t \in \mathbb{R} \right\}$.

So to find all of the solutions of the system we only need to find a single solution.

To find a single solution just let $x_3 = 0$, for example, then the system becomes

$$2x_1 - x_2 = 6$$

$$\underline{x_1 + x_2 = 9}$$

$$3x_1 \qquad = 15 \quad \Longrightarrow \quad x_1 = 5, \quad x_2 = 4.$$

Thus $< 5,4,0 >$ is a solution. Hence the set of all solutions is

$$K = \left\{ < 5,4,0 > + t < 1, \frac{1}{2}, -\frac{3}{2} > \middle| \ t \in \mathbb{R} \right\}.$$

Ex. Consider the system $2x_1 - x_2 + x_3 = 8$. Find all solutions.

We found earlier that $K_H = \{t_1 < 0,1,1 > +t_2 < 1,2,0 >| \, t_1, t_2 \in \mathbb{R}\}$.

So we just need to find a single solution of $2x_1 - x_2 + x_3 = 8$.

If we let $x_1 = x_2 = 0$, then $x_3 = 8$.

Thus $< 0,0,8 >$ is a solution. Hence all of the solutions are given by:

$$K = \{< 0,0,8 > +t_1 < 0,1,1 > +t_2 < 1,2,0 >| \; t_1, t_2 \in \mathbb{R}\}.$$

Theorem: Let $Ax = b$ be a system of n linear equations in n unknowns. If A is invertible, then the system has exactly one solution, $A^{-1}b$. Conversely, if the system has exactly one solution then A is invertible.

Proof: If A is invertible then

$$Ax = b$$

$$A^{-1}Ax = A^{-1}b$$

$$Ix = A^{-1}b$$

$$x = A^{-1}b.$$

Now suppose the system has exactly one solution, s.

But we know that the solution set is $\{s\} = \{s\} + K_H \implies \quad K_H = \{0\}$.

Hence $N(L_A) = \{0\}$ and A is invertible.

Ex. Consider the system of equations:

$$\begin{aligned} x_1 + 4x_2 + 3x_3 &= 12 \\ -x_1 - 2x_2 \qquad &= 4 \\ 2x_1 + 2x_2 + 3x_3 &= 24. \end{aligned}$$

Find all of the solutions.

In matrix form $Ax = b$ we have:

$$\begin{bmatrix} 1 & 4 & 3 \\ -1 & -2 & 0 \\ 2 & 2 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 12 \\ 4 \\ 24 \end{bmatrix}.$$

In an example in the previous section we found A^{-1} to be:

$$A^{-1} = \begin{bmatrix} -\frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \\ \frac{1}{4} & -\frac{1}{4} & -\frac{1}{4} \\ \frac{1}{6} & \frac{1}{2} & \frac{1}{6} \end{bmatrix}.$$

Thus there is a unique solution, $A^{-1}\begin{bmatrix} 12 \\ 4 \\ 24 \end{bmatrix}$:

$$\begin{bmatrix} -\frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \\ \frac{1}{4} & -\frac{1}{4} & -\frac{1}{4} \\ \frac{1}{6} & \frac{1}{2} & \frac{1}{6} \end{bmatrix} \begin{bmatrix} 12 \\ 4 \\ 24 \end{bmatrix} = \begin{bmatrix} 4 \\ -4 \\ 8 \end{bmatrix}.$$

So the unique solution to this system is $< 4, -4, 8 >$.

Ex. Suppose that A is a 3×5 matrix representing a linear transformation $T: \mathbb{R}^5 \to \mathbb{R}^3$ with $\dim(N(T)) = 2$. Show that for any $b \in \mathbb{R}^3$

a. $Ax = b$ has at least one solution $x \in \mathbb{R}^5$, ie, $b \in R(T)$.

b. $Ax = b$ has an infinite number of solutions.

a. By the dimension theorem we have:

$$\dim(N(T)) + \dim(R(T)) = \dim(\mathbb{R}^5)$$
$$2 \qquad + \dim(R(T)) = 5$$
$$\dim(R(T)) = 3.$$

But $\dim(\mathbb{R}^3) = 3 = \dim(R(T))$ so T (and A) is onto.

Hence given any $b \in \mathbb{R}^3$ there is at least one solution $x \in \mathbb{R}^5$ with $Ax = b$.

b. All solutions of $Ax = b$ are of the form $x_p + x_0$, where x_p is any particular solution to $Ax = b$, and x_0 is any solutions of $Ax = 0$.

But all solutions of $Ax = 0$ are precisely the elements of the set $N(T)$.

Since $\dim(N(T)) = 2$ we know that we can write:

$$N(T) = \{av_1 + bv_2 | \; a, b \in \mathbb{R}\}, \quad \text{where } v_1, v_2 \text{ are basis vectors.}$$

Hence $N(T)$ contains an infinity number of elements.

By part a we know that there is at least one particular solution of $Ax = b$.

Hence all solutions of $Ax = b$, $\{x_p + x_0\}$, is an infinite set.