A Matrix's Rank and Calculating Inverse Matrices

Def. If $A \in M_{m\times n}(\mathbb{R})$, we define the **rank of $\boldsymbol{A}$**, denoted $Rank(A)$, to be the rank of the linear transformation associated with A, L_A, $L_A: \mathbb{R}^n \to \mathbb{R}^m$.

If $m = n$, then notice that an $n \times n$ matrix is invertible if and only if its rank is n. This follows from an earlier theorem about linear transformations. This is because any matrix A is the matrix representation of a linear transformation. In fact we have:

Theorem: Let $T: V \to W$ be a linear transformation between finite dimensional vector spaces, and let B_1 and B_2 be ordered bases for V and W respectively. Then $Rank(T) = Rank\left([T]_{B_1}^{B_2}\right)$.

Theorem: Let A be an $m \times n$ matrix. If P is an $m \times m$ matrix and Q is an $n \times n$ matrix, both invertible, then

a. $Rank(AQ) = Rank(A)$

b. $Rank(PA) = Rank(A)$

c. $Rank(PAQ) = Rank(A)$.

Proof: a. $R(L_{AQ}) = R(L_A L_Q) = L_A L_Q(\mathbb{R}^n)$

$$= L_A\left(L_Q(\mathbb{R}^n)\right) = L_A(\mathbb{R}^n) \qquad \text{(since } L_Q \text{ is onto)}$$

$$= R(L_A).$$

Thus we have:

$$Rank(AQ) = \dim\left(R(L_{AQ})\right) = \dim\left(R(L_A)\right) = Rank(A).$$

b. Similarly, we have:

$$R(L_{PA}) = R\big(L_P(L_A)\big) = L_P(L_A(\mathbb{R}^n)).$$

But $L_A(\mathbb{R}^n)$ is a subspace of R^m.

Since P is invertible we have:

$$\dim\,(L_P\big(L_A(\mathbb{R}^n)\big) = \dim\big(L_A(\mathbb{R}^n)\big) = Rank(A).$$

So the $Rank(PA) = Rank(A)$.

c. Follows from parts a and b.

Corollary: Elementary row and column operations on a matrix are rank preserving.

Proof: Every elementary row or column operation can be viewed as a multiplicaion of a matrix by an invertible matrix on the left (elementary row operations) or the right (elementary column operations).

Theorem: The rank of any matrix equals the maximum number of its linearly independent columns. Thus the rank of a matrix is the dimension of the subspace generated by its columns.

Proof: Let $A \in M_{m\times n}(\mathbb{R})$.

$$Rank(A) = Rank(L_A) = \dim\big(R(L_A)\big).$$

Let B be the standard ordered basis for $\mathbb{R}^n$.

Then we have:

$$R(L_A) = span\{L_A(B)\}$$
$$= span\{L_A(e_1), \dots, L_A(e_n)\}$$

But $L_A(e_j) = j^{th}$ column of $A = v_j$.

Thus $R(L_A) = span\{v_1, \dots, v_n\}$.

Hence $Rank(A) = \dim\big(R(L_A)\big) = \dim(span\{v_1, \dots, v_n\})$.

Ex. Let $A = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 2 \\ 1 & 0 & 1 \end{bmatrix}$. Find the $Rank(A)$.

Notice that columns one and two are linearly independent, but column 3 is the sum of columns one and two. Thus we have:

$$Rank(A) = \dim\left(Span\left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} \right\} \right) = 2.$$

In general it can be difficult to identify the maximum number of linearly independent columns of a matrix A. However, we know that we don't change $Rank(L_A)$ by performing elementary row (or column) operations (since they are invertible). Thus we can find the maximum number of linearly independent columns of a matrix A through elementary row and column operations.

Ex. Let $A = \begin{bmatrix} 1 & 2 & 1 \\ 2 & 1 & 5 \\ 2 & 3 & 3 \end{bmatrix}$. Find $Rank(A)$.

Subtracting 2(row 1) from row 2 and replacing it in row 2: $R_2 - 2R_1 \rightarrow R_2$.

$$\begin{bmatrix} 1 & 2 & 1 \\ 2 & 1 & 5 \\ 2 & 3 & 3 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 2 & 1 \\ 0 & -3 & 3 \\ 2 & 3 & 3 \end{bmatrix}.$$

Now subtract 2(row 1) from row 3 and replace it in row 3: $R_3 - 2R_1 \rightarrow R_3$.

$$\begin{bmatrix} 1 & 2 & 1 \\ 0 & -3 & 3 \\ 2 & 3 & 3 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 2 & 1 \\ 0 & -3 & 3 \\ 0 & -1 & 1 \end{bmatrix}.$$

Next subtract 2(column 1) from column 2 and replace it in column 2:
$C_2 - 2C_1 \rightarrow C_2$.

$$\begin{bmatrix} 1 & 2 & 1 \\ 0 & -3 & 3 \\ 0 & -1 & 1 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 1 \\ 0 & -3 & 3 \\ 0 & -1 & 1 \end{bmatrix}.$$

Finally, subtract column 1 from column 3 and replace it in column 3: $C_3 - C_1 \rightarrow C_3$

$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & -3 & 3 \\ 0 & -1 & 1 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & -3 & 3 \\ 0 & -1 & 1 \end{bmatrix}.$$

It's now clear that $C_3 = -C_2$ and the C_1 and C_2 are linearly independent. Thus $Rank(A) = 2$.

In fact, given an $m \times n$ matrix A we can always transform it using elementary row and column operations into a matrix that looks like:

$$D = \begin{bmatrix} I_r & 0_1 \\ 0_2 & 0_3 \end{bmatrix}$$

where $0_1, 0_2$, and 0_3 are zero matrices.

Ex. Put $A = \begin{bmatrix} 0 & 2 & 2 & 0 \\ 2 & 2 & 0 & 4 \\ 4 & 6 & 4 & 6 \\ 0 & 4 & 2 & 2 \end{bmatrix}$ in the form $D = \begin{bmatrix} I_r & 0_1 \\ 0_2 & 0_3 \end{bmatrix}$ through elementary row and column operations and find the $Rank(A)$.

$$\begin{bmatrix} 0 & 2 & 2 & 0 \\ 2 & 2 & 0 & 4 \\ 4 & 6 & 4 & 6 \\ 0 & 4 & 2 & 2 \end{bmatrix} \xrightarrow[R_2 \leftrightarrow R_1]{} \begin{bmatrix} 2 & 2 & 0 & 4 \\ 0 & 2 & 2 & 0 \\ 4 & 6 & 4 & 6 \\ 0 & 4 & 2 & 2 \end{bmatrix} \xrightarrow[\frac{1}{2}R_1 \to R_1]{} \begin{bmatrix} 1 & 1 & 0 & 2 \\ 0 & 2 & 2 & 0 \\ 4 & 6 & 4 & 6 \\ 0 & 4 & 2 & 2 \end{bmatrix}$$

$$\xrightarrow[R_3 - 4R_1 \to R_3]{} \begin{bmatrix} 1 & 1 & 0 & 2 \\ 0 & 2 & 2 & 0 \\ 0 & 2 & 4 & -2 \\ 0 & 4 & 2 & 2 \end{bmatrix} \xrightarrow[C_2 - C_1 \to C_2]{} \begin{bmatrix} 1 & 0 & 0 & 2 \\ 0 & 2 & 2 & 0 \\ 0 & 2 & 4 & -2 \\ 0 & 4 & 2 & 2 \end{bmatrix}$$

$$\xrightarrow[R_3 - R_2 \to R_3]{} \begin{bmatrix} 1 & 0 & 0 & 2 \\ 0 & 2 & 2 & 0 \\ 0 & 0 & 2 & -2 \\ 0 & 4 & 2 & 2 \end{bmatrix} \xrightarrow[R_4 - 2R_2 \to R_4]{} \begin{bmatrix} 1 & 0 & 0 & 2 \\ 0 & 2 & 2 & 0 \\ 0 & 0 & 2 & -2 \\ 0 & 0 & -2 & 2 \end{bmatrix}$$

$$\xrightarrow{R_3+R_4\to R_4}\begin{bmatrix}1&0&0&2\\0&2&2&0\\0&0&2&-2\\0&0&0&0\end{bmatrix}\xrightarrow{\frac{1}{2}R_3\to R_3}\begin{bmatrix}1&0&0&2\\0&2&2&0\\0&0&1&-1\\0&0&0&0\end{bmatrix}$$

$$\xrightarrow{\frac{1}{2}R_2\to R_2}\begin{bmatrix}1&0&0&2\\0&1&1&0\\0&0&1&-1\\0&0&0&0\end{bmatrix}\xrightarrow{C_4-2C_1\to C_4}\begin{bmatrix}1&0&0&0\\0&1&1&0\\0&0&1&-1\\0&0&0&0\end{bmatrix}$$

$$\xrightarrow{C_3-C_2\to C_3}\begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&1&-1\\0&0&0&0\end{bmatrix}\xrightarrow{C_3+C_4\to C_4}\begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&0\end{bmatrix}.$$

Thus the $Rank(A) = 3$.

Note: One does not necessarily need to transform a matrix into the form $D = \begin{bmatrix}I_r & 0_1\\0_2 & 0_3\end{bmatrix}$ to identify its rank.

Ex. Find the rank of $A = \begin{bmatrix}1&3&1&1\\2&0&2&2\\1&6&1&1\end{bmatrix}$.

$$\begin{bmatrix}1&3&1&1\\2&0&2&2\\1&6&1&1\end{bmatrix}\xrightarrow{R_2-2R_1\to R_2}\begin{bmatrix}1&3&1&1\\0&-6&0&0\\1&6&1&1\end{bmatrix}\xrightarrow{R_3-R_1\to R_3}\begin{bmatrix}1&3&1&1\\0&-6&0&0\\0&3&0&0\end{bmatrix}.$$

It's now clear that there are two linearly independent column vectors so $Rank(A) = 2$.

Ex. Find the rank of $A = \begin{bmatrix} 1 & 2 & 3 & 1 \\ 2 & 1 & 1 & 1 \\ 1 & -1 & 1 & 0 \end{bmatrix}$.

$$\begin{bmatrix} 1 & 2 & 3 & 1 \\ 2 & 1 & 1 & 1 \\ 1 & -1 & 1 & 0 \end{bmatrix} \xrightarrow[R_2 - 2R_1 \to R_2]{} \begin{bmatrix} 1 & 2 & 3 & 1 \\ 0 & -3 & -5 & -1 \\ 1 & -1 & 1 & 0 \end{bmatrix} \xrightarrow[R_3 - R_1 \to R_3]{} \begin{bmatrix} 1 & 2 & 3 & 1 \\ 0 & -3 & -5 & -1 \\ 0 & -3 & -2 & -1 \end{bmatrix}$$

$$\xrightarrow[R_3 - R_2 \to R_3]{} \begin{bmatrix} 1 & 2 & 3 & 1 \\ 0 & -3 & -5 & -1 \\ 0 & 0 & 3 & 0 \end{bmatrix}.$$

It is now clear that $< 1,0,0 >, < 2,-3,0 >$ and $< 3,-5,3 >$ are linearly independent in $\mathbb{R}^3$. Since one can have at most 3 linearly independent vectors in $\mathbb{R}^3$, the $Rank(A) = 3$.

Given any $n \times n$ matrix A we can put it in the form:

$$D = \begin{bmatrix} I_r & 0_1 \\ 0_2 & 0_3 \end{bmatrix}$$

using elementary row and column operations. In particular, if A is invertible then

$$D = \begin{bmatrix} 1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 1 \end{bmatrix} = I_n.$$

So if A is $n \times n$ and invertible then there exist invertible matrices B and C such that: $$I_n = BAC$$
where $B = E_p \cdots E_1$ and $C = G_1 \cdots G_q$ are products of elementary matrices.

But if we have $I_n = BAC$ then we have:

$$\begin{aligned} I_n &= BAC \\ I_n C^{-1} &= BACC^{-1} \\ C^{-1} &= BA \\ CC^{-1} &= CBA \\ I_n &= CBA. \end{aligned}$$

Thus we can write:

$$E_1 \cdots E_k A = I_n$$

where the $E_i{}'s$ are elementary matrices.

But then we have:

$$\begin{aligned} E_1 \cdots E_k A A^{-1} &= I_n A^{-1} = A^{-1} \\ E_1 \cdots E_k I_n &= A^{-1}. \end{aligned}$$

Thus to find A^{-1} we just need to apply to I_n the same elemtary row operations that turned A into I_n.

Ex. Let $A = \begin{bmatrix} 1 & 4 & 3 \\ -1 & -2 & 0 \\ 2 & 2 & 3 \end{bmatrix}$. Find A^{-1}.

We start by creating the augmented matrix $(A|I_3)$.

We will then apply a sequence of elementary row operations that transform A into I_3, to both A and I_3.

<u>Row Operations</u>

$$\left[\begin{array}{ccc|ccc} 1 & 4 & 3 & 1 & 0 & 0 \\ -1 & -2 & 0 & 0 & 1 & 0 \\ 2 & 2 & 3 & 0 & 0 & 1 \end{array}\right] \rightarrow \left[\begin{array}{ccc|ccc} 1 & 4 & 3 & 1 & 0 & 0 \\ -1 & -2 & 0 & 0 & 1 & 0 \\ 0 & -6 & -3 & -2 & 0 & 1 \end{array}\right] \qquad R_3 - 2R_1 \rightarrow R_3$$

$$\rightarrow \left[\begin{array}{ccc|ccc} 1 & 4 & 3 & 1 & 0 & 0 \\ 0 & 2 & 3 & 1 & 1 & 0 \\ 0 & -6 & -3 & -2 & 0 & 1 \end{array}\right] \qquad R_1 + R_2 \rightarrow R_2$$

$$\rightarrow \left[\begin{array}{ccc|ccc} 1 & 4 & 3 & 1 & 0 & 0 \\ 0 & 2 & 3 & 1 & 1 & 0 \\ 0 & 0 & 6 & 1 & 3 & 1 \end{array}\right] \qquad R_3 + 3R_2 \rightarrow R_3$$

$$\rightarrow \left[\begin{array}{ccc|ccc} 1 & 2 & 0 & 0 & -1 & 0 \\ 0 & 2 & 3 & 1 & 1 & 0 \\ 0 & 0 & 6 & 1 & 3 & 1 \end{array}\right] \qquad R_1 - R_2 \rightarrow R_1$$

$$\rightarrow \left[\begin{array}{ccc|ccc} 1 & 2 & 0 & 0 & -1 & 0 \\ 0 & 2 & 0 & \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} \\ 0 & 0 & 6 & 1 & 3 & 1 \end{array}\right] \qquad R_2 - \frac{1}{2}R_3 \rightarrow R_2$$

$$\rightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -\frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \\ 0 & 2 & 0 & \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} \\ 0 & 0 & 6 & 1 & 3 & 1 \end{array}\right] \qquad R_1 - R_2 \rightarrow R_1$$

$$\rightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -\frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \\ 0 & 1 & 0 & \frac{1}{4} & -\frac{1}{4} & -\frac{1}{4} \\ 0 & 0 & 1 & \frac{1}{6} & \frac{1}{2} & \frac{1}{6} \end{array}\right] \qquad \frac{1}{6}R_3 \rightarrow R_3,\ \frac{1}{2}R_2 \rightarrow R_2$$

So $A^{-1} = \begin{bmatrix} -\frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \\ \frac{1}{4} & -\frac{1}{4} & -\frac{1}{4} \\ \frac{1}{6} & \frac{1}{2} & \frac{1}{6} \end{bmatrix}$. A straight forward calculation will show that $AA^{-1} = A^{-1}A = I_3$.