Elementary Matrices

Def. Let A be an $m \times n$ matrix. Any of the following are called elementary row (column) operations.

1. Interchanging two rows (columns) of A.
2. Multiply a row (column) by a nonzero real number.
3. Replace a row (column) by its sum with a multiple of another row (column).

Ex. Let $A = \begin{bmatrix} 1 & -2 & 3 & 5 \\ 2 & -3 & 4 & 1 \\ 6 & 0 & 1 & 2 \end{bmatrix}$

a. An example of an elementary row operation of type 1 on A:

interchange rows 1 and 3 which we will denote $R_1 \leftrightarrow R_3$.

$$B = \begin{bmatrix} 6 & 0 & 1 & 2 \\ 2 & -3 & 4 & 1 \\ 1 & -2 & 3 & 5 \end{bmatrix}$$

b. An example of an elementary row operation of type 2 on A:

multiply the second row of A by -2, denoted $-2R_2 \rightarrow R_2$.

$$C = \begin{bmatrix} 1 & -2 & 3 & 5 \\ -4 & 6 & -8 & -2 \\ 6 & 0 & 1 & 2 \end{bmatrix}$$

c. An example of an elementary row operation of type 3 on A:

replace row one with row one minus 2 times row three, denoted $R_1 - 2R_3 \rightarrow R_1$.

$$D = \begin{bmatrix} -11 & -2 & 1 & 1 \\ 2 & -3 & 4 & 1 \\ 6 & 0 & 1 & 2 \end{bmatrix}.$$

Notice that if a matrix M can be obtained from a matrix P by an elementary row (column) operation then matrix P can be obtained from matrix M by an elementary row (column) operation.

Ex. For a 3×3 matrix there are only three possibilities for an elementary row operation of type 1, interchange rows one and two, interchange rows one and three, or interchange rows two and three. If we apply those operations to the identity matrix we get:

$$E_1 = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \qquad E_1 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \quad \text{or} \quad E_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}.$$

Notice that if we take any 3×3 matrix A and multiply it on the left by any of the E_1's, it will interchange the corresponding rows of A. For example, if

$$A = \begin{bmatrix} 1 & 4 & 3 \\ -1 & -2 & 0 \\ 2 & 2 & 3 \end{bmatrix}$$ and we multiply it on the left by

$$E_1 = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$ we get:

$$E_1 A = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 4 & 3 \\ -1 & -2 & 0 \\ 2 & 2 & 3 \end{bmatrix} = \begin{bmatrix} -1 & -2 & 0 \\ 1 & 4 & 3 \\ 2 & 2 & 3 \end{bmatrix}.$$

So multiplying A on the left by E_1 interchanges rows one and two of A.

Ex. An elementary row operation of type 2 is obtained by multiplying a row of I_n by a nonzero real number. Thus for a 3×3 matrix we get:

$$E_2 = \begin{bmatrix} \alpha & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \qquad E_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \beta & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad \text{or} \quad E_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \gamma \end{bmatrix}$$

where α, β, and γ are nonzero real numbers.

Notice that if we multiply $A = \begin{bmatrix} 1 & 4 & 3 \\ -1 & -2 & 0 \\ 2 & 2 & 3 \end{bmatrix}$ on the left by $E_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

it will multiply the second row of A by 3:

$$E_2A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 4 & 3 \\ -1 & -2 & 0 \\ 2 & 2 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 4 & 3 \\ -3 & -6 & 0 \\ 2 & 2 & 3 \end{bmatrix}.$$

Ex. An elementary row operation of type 3 is obtained from I_n by adding a multiple of one row to another. For a 3×3 matrix there are 6 possibilities.

$a_1R_1 + R_2 \to R_2$ $\quad$ $a_2R_1 + R_3 \to R_3$ $\quad$ $a_3R_2 + R_1 \to R_1$

$$E_3 = \begin{bmatrix} 1 & 0 & 0 \\ a_1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad E_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ a_2 & 0 & 1 \end{bmatrix}, \quad E_3 = \begin{bmatrix} 1 & a_3 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

$a_4R_2 + R_3 \to R_3$ $\quad$ $a_5R_3 + R_1 \to R_1$ $\quad$ $a_6R_3 + R_2 \to R_2$

$$E_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & a_4 & 1 \end{bmatrix}, \quad E_3 = \begin{bmatrix} 1 & 0 & a_5 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad \text{and} \quad E_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & a_6 \\ 0 & 0 & 1 \end{bmatrix},$$

where $a_1, a_2, \ldots a_6$ are nonzero real numbers.

Notice that if we multiply a 3×3 matrix A on the left by E_3, it will multiply a row of A by a constant and add it to another row. For example, if we multiply A by $E_3 = \begin{bmatrix} 1 & 4 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, it will multiply the second row of A by 4 and add it to the first row.

$$E_3A = \begin{bmatrix} 1 & 4 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 4 & 3 \\ -1 & -2 & 0 \\ 2 & 2 & 3 \end{bmatrix} = \begin{bmatrix} -3 & -4 & 3 \\ -1 & -2 & 0 \\ 2 & 2 & 3 \end{bmatrix}.$$

Thus every elementary row operation done to an $m \times n$ matrix A can be done as a multiplication of A on the left by an elementary $m \times m$ matrix E_i, $\;i = 1,2,3$.

Theorem: If E is an elementary matrix, then E is invertible and E^{-1} is an elementary matrix of the same type.

Proof: (We give the argument for elementary row operations. Similar arguments work for elementary column operations).

If E is an elementary matrix of type 1, it interchanges two rows of I_n.

But then doing the same operation a second time will bring the matrix back to I_n.

Thus $EE = I_n$, so $E^{-1} = E$ and E^{-1} is type 1.

If E is an elementary matrix of type 2, then it multiplies a row of I_n by some nonzero number m. Then E^{-1} is just the type 2 elementary matrix that multiplies the same row by $\frac{1}{m}$.

If E is an elementary matrix of type 3, it is obtained by multiplying the i^{th} row of I_n by a nonzero number m and adding it to the j^{th} row. E^{-1} is obtained by multiplying the i^{th} row by $-m$ and adding it to the j^{th} row. E^{-1} is clearly a type 3 elementary matrix.

When we multiply an $m \times n$ matrix on the right by an elementary $n \times n$ matrix we get the corresponding elementary column operations.

Ex. If $E_1 = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, which is what we get if we switch the first two columns of $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, and $A = \begin{bmatrix} 1 & 4 & 3 \\ -1 & -2 & 0 \\ 2 & 2 & 3 \end{bmatrix}$, then

$$AE_1 = \begin{bmatrix} 1 & 4 & 3 \\ -1 & -2 & 0 \\ 2 & 2 & 3 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 4 & 1 & 3 \\ -2 & -1 & 0 \\ 2 & 2 & 3 \end{bmatrix}.$$

Thus multiplying A by E_1 on the right interchanges the first two columns.