# Riemannian Metrics

Def. Let $M$ be a smooth manifold. A **Riemannian metric** on $M$ is a symmetric bilinear form at each point $p \in M$ that takes elements of $(X, Y) \in T_pM \times T_pM$ into a real number $g(X, Y)$ and $g(X, X) > 0$ if $X \neq 0$ (i.e. it's positive definite).

Def. A smooth manifold, $M$, together with a Riemannian metric, $g$, is called a **Riemannian manifold**, $(M, g)$.

Ex. If $M = \mathbb{R}^2$ and $g = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, then $g(X, Y)$ is the standard Euclidean inner product on $\mathbb{R}^2$.

$$X = (\alpha_1, \alpha_2); \qquad Y = (\beta_1, \beta_2)$$

$$< X, Y > = g(X, Y) = (\alpha_1 \quad \alpha_2) \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \beta_1 \\ \beta_2 \end{pmatrix} = \alpha_1\beta_1 + \alpha_2\beta_2 \, .$$

However, if we let $\bar{g}$ be represented by $\bar{g} = \begin{pmatrix} 2 & 1 \\ 1 & 3 \end{pmatrix}$ we would get a different inner product on $\mathbb{R}^2$:

$$< X, Y > = \bar{g}(X, Y) = (\alpha_1 \quad \alpha_2) \begin{pmatrix} 2 & 1 \\ 1 & 3 \end{pmatrix} \begin{pmatrix} \beta_1 \\ \beta_2 \end{pmatrix}$$

$$= 2\alpha_1\beta_1 + \alpha_1\beta_2 + \alpha_2\beta_1 + 3\alpha_2\beta_2.$$

Def. Let $(M, g)$ be a Riemannian manifold. Suppose that $X$ and $Y$ are vectors in $T_pM$.

1. The **length of $\boldsymbol{X}$**, denoted $\|X\|$, is defined as $\|X\| = \sqrt{g(X, X)}$
2. The **angle $\boldsymbol{\theta}$ between $\boldsymbol{X}$ and $\boldsymbol{Y}$** is defined by $\cos\theta = \frac{g(X,Y)}{\|X\| \, \|Y\|}$
3. $X$ and $Y$ are called **orthogonal** if $g(X, Y) = 0$.

Two Riemannian manifolds are considered the same if they have the same metric.

Def. Let $M$ and $N$ be two Riemannian manifolds. A diffeomorphism $f\colon M \to N$ is called an **isometry** if for all $p \in M$ the following holds:
$< X, Y >_p = < df_p(X), df_p(Y) >_{f(p)}$ for $X, Y \in T_pM$.

Two Riemannian manifolds are called **isometric** if there exists an isometry between them.

Given a parametrization of a manifold $\overrightarrow{\Phi}$, there is a Riemannian metric associated with $\overrightarrow{\Phi}$. We call this the **metric induced by $\overrightarrow{\boldsymbol{\Phi}}$.**

Ex. Let $\overrightarrow{\Phi}(u, v) = (u, v, u^2 + v^2)$ be a parametrization of a surface $S$. Find the metric induce by $\overrightarrow{\Phi}$.

At $(u, v) = (1, 2)$, Let $X \in T_{(1,2,5)}S$ be given by $< 2, -3 >$, i.e., $X = 2\overrightarrow{\Phi}_u(1, 2) - 3\overrightarrow{\Phi}_v(1, 2)$. Find $\|X\|$?

$$\overrightarrow{\Phi}_u = < 1, 0, 2u >, \qquad \overrightarrow{\Phi}_v = < 0, 1, 2v >$$

As we saw earlier, we can get the metric induced by $\overrightarrow{\Phi}$ on $S$ by:

$$g_{11} = \overrightarrow{\Phi}_u \cdot \overrightarrow{\Phi}_u = 1 + 4u^2$$
$$g_{12} = g_{21} = \overrightarrow{\Phi}_u \cdot \overrightarrow{\Phi}_v = 4uv$$
$$g_{22} = \overrightarrow{\Phi}_v \cdot \overrightarrow{\Phi}_v = 1 + 4v^2$$

So the metric induced by $\overrightarrow{\Phi}$ is given by:

$$g = \begin{pmatrix} 1 + 4u^2 & 4uv \\ 4uv & 1 + 4v^2 \end{pmatrix}$$

with respect to the basis $\{\overrightarrow{\Phi}_u, \overrightarrow{\Phi}_v\} = \{< 1, 0, 2u >, \ < 0, 1, 2v >\}$.

$$\|X\| = \sqrt{g(X,X)}$$

At $(u,v) = (1,2)$, $\quad g = \begin{pmatrix} 5 & 8 \\ 8 & 17 \end{pmatrix}$.

$X =< 2, -3 >$, so we can write:

$$g(X,X) = (2 \quad -3)\begin{pmatrix} 5 & 8 \\ 8 & 17 \end{pmatrix}\begin{pmatrix} 2 \\ -3 \end{pmatrix} = (2 \quad -3)\begin{pmatrix} -14 \\ -35 \end{pmatrix} = 77.$$

So $\|X\| = \sqrt{77}$ .

Notice that $X$ is also a vector in $T_{(1,2,5)}\mathbb{R}^3$. What is $\|X\|$ with respect to the Euclidean metric in $\mathbb{R}^3$?

$\vec{\Phi}_u(1,2) =< 1, 0, 2 >$ and $\vec{\Phi}_v(1,2) =< 0, 1, 4 >$.

$$X = 2\vec{\Phi}_u(1,2) - 3\vec{\Phi}_v(1,2) = 2 < 1,0,2 > -3 < 0,1,4 >$$
$$=< 2, -3, -8 >.$$

In $\mathbb{R}^3$ the Euclidean metric is:

$$\bar{g} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\bar{g}(X,X) = (2 \quad -3 \quad -8)\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 2 \\ -3 \\ -8 \end{pmatrix}$$

$$=< 2, -3, -8 > \cdot < 2, -3, -8 > = 4 + 9 + 64 = 77$$

So $\|X\|_{\bar{g}} = \sqrt{77}$ (the same as before!).

The length of a vector using the induced metric will always be the same as the length from the Euclidean metric.

Let $X = \alpha\overrightarrow{\Phi}_u + \beta\overrightarrow{\Phi}_v$, and $(g_{ij})$ the induced metric.
Then we have:

$$\begin{aligned} g(X,X) &= (\alpha \quad \beta)\begin{pmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{pmatrix}\begin{pmatrix} \alpha \\ \beta \end{pmatrix} \\ &= \alpha^2 g_{11} + 2\alpha\beta g_{12} + \beta^2 g_{22}. \end{aligned}$$

Now let's use the Euclidean inner product to find $\|X\|^2$ :

$$\begin{aligned} X \cdot X &= (\alpha\overrightarrow{\Phi}_u + \beta\overrightarrow{\Phi}_v) \cdot (\alpha\overrightarrow{\Phi}_u + \beta\overrightarrow{\Phi}_v) \\ &= \alpha^2\overrightarrow{\Phi}_u \cdot \overrightarrow{\Phi}_u + 2\alpha\beta\overrightarrow{\Phi}_u \cdot \overrightarrow{\Phi}_v + \beta^2\overrightarrow{\Phi}_v \cdot \overrightarrow{\Phi}_v \\ &= \alpha^2 g_{11} + 2\alpha\beta g_{12} + \beta^2 g_{22} \\ &= g(X,X). \end{aligned}$$

In fact, the above analysis leads us to why the calculation of the induced metric gives us a matrix that is positive definite.

Suppose $\overrightarrow{\Phi}: U \subseteq \mathbb{R}^n \to M \subseteq \mathbb{R}^k$ is a parametrization of an $n$-dimensional manifold $M$, with local coordinates $(x^1, \dots, x^n)$. The induced metric from $\overrightarrow{\Phi}$ is:

$$g = (g_{ij}), \quad \text{where } g_{ij} = \overrightarrow{\Phi}_{x^i} \cdot \overrightarrow{\Phi}_{x^j}.$$

We can represent any vector $X \in T_pM$ by:

$$X = a_1\overrightarrow{\Phi}_{x^1} + \cdots + a_n\overrightarrow{\Phi}_{x^n}.$$

Thus we can say if $X \neq \vec{0}$:

$$\begin{aligned} 0 < X \cdot X &= (a_1\overrightarrow{\Phi}_{x^1} + \cdots + a_n\overrightarrow{\Phi}_{x^n}) \cdot (a_1\overrightarrow{\Phi}_{x^1} + \cdots + a_n\overrightarrow{\Phi}_{x^n}) \\ &= \textstyle\sum_{i,j=1}^{n} \left(\overrightarrow{\Phi}_{x^i} \cdot \overrightarrow{\Phi}_{x^j}\right)(a_i a_j) \\ &= \textstyle\sum_{i,j=1}^{n} g_{ij}(a_i a_j) \\ &= g(X,X). \end{aligned}$$

Ex. Let $S^2_+ \subseteq \mathbb{R}^3$ be the upper hemisphere and $S^2_- \subseteq \mathbb{R}^3$ be the lower hemisphere parametrized by:

$$\vec{\Phi}(u,v) = \left(u, v, \sqrt{1-u^2-v^2}\right) \ , \qquad u^2+v^2<1$$

$$\vec{\Psi}(\bar{u},\bar{v}) = \left(\bar{u}, \bar{v}, -\sqrt{1-\bar{u}^2-\bar{v}^2}\right) \ , \qquad \bar{u}^2+\bar{v}^2<1$$

Let $f: S^2_+ \to S^2_-$ by:

$$f\left(u, v, \sqrt{1-u^2-v^2}\right) = \left(-v, u, -\sqrt{1-u^2-v^2}\right)$$

Show that $f$ is an isometry.

Showing that $f$ is a diffeomorphism is fairly straight-forward. Let's show that:

$$< X, Y >_p = < df_p(X), df_p(Y) >_{f(p)}$$

Let's start by calculating the metric on $S^2_+$ induced by $\vec{\Phi}$:

$$\vec{\Phi}_u = \left(1, 0, -\frac{u}{\sqrt{1-u^2-v^2}}\right) \text{ and } \vec{\Phi}_v = \left(0, 1, -\frac{v}{\sqrt{1-u^2-v^2}}\right)$$

$$g_{11} = \vec{\Phi}_u \cdot \vec{\Phi}_u = 1 + \frac{u^2}{1-u^2-v^2} = \frac{1-v^2}{1-u^2-v^2}$$

$$g_{12} = g_{21} = \vec{\Phi}_u \cdot \vec{\Phi}_v = \frac{uv}{1-u^2-v^2}$$

$$g_{22} = \vec{\Phi}_v \cdot \vec{\Phi}_v = 1 + \frac{v^2}{1-u^2-v^2} = \frac{1-u^2}{1-u^2-v^2} \ .$$

So given the vectors $X, Y \in T_p S_+^2$

$$X = \alpha_1 \vec{\Phi}_u + \alpha_2 \vec{\Phi}_v =< \alpha_1, \alpha_2 >$$
$$Y = \beta_1 \vec{\Phi}_u + \beta_2 \vec{\Phi}_v =< \beta_1, \beta_2 >$$

$$< X, Y >_p = g(X, Y) = (\alpha_1 \quad \alpha_2) \begin{pmatrix} \frac{1-v^2}{1-u^2-v^2} & \frac{uv}{1-u^2-v^2} \\ \frac{uv}{1-u^2-v^2} & \frac{1-u^2}{1-u^2-v^2} \end{pmatrix} \begin{pmatrix} \beta_1 \\ \beta_2 \end{pmatrix}$$

$$< X, Y >_p = \frac{\alpha_1\beta_1(1-v^2)+(\alpha_1\beta_2+\alpha_2\beta_1)uv+\alpha_2\beta_2(1-u^2)}{1-u^2-v^2}.$$

Notice that:

$\vec{\Psi}^{-1} \circ f \circ \vec{\Phi}(u, v) = \vec{\Psi}^{-1}\left(-v, u, -\sqrt{1 - u^2 - v^2}\right) = (-v, u) = (\bar{u}, \bar{v})$.

So $\bar{u} = -v$ and $\bar{v} = u$.

Thus:

$$df_p = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

$$df_p(X) = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \alpha_1 \\ \alpha_2 \end{pmatrix} = (-\alpha_2, \alpha_1) = -\alpha_2 \Psi_{\bar{u}} + \alpha_1 \Psi_{\bar{v}}$$

$$df_p(Y) = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \beta_1 \\ \beta_2 \end{pmatrix} = (-\beta_2, \beta_1) = -\beta_2 \Psi_{\bar{u}} + \beta_1 \Psi_{\bar{v}}.$$

Now we need the induced metric $\bar{g}$ on $S^2_-$ from $\vec{\Psi}$.

$$\vec{\Psi}_{\bar{u}} = \left(1, 0, \frac{\bar{u}}{\sqrt{1-\bar{u}^2-\bar{v}^2}}\right) \text{ and } \quad \vec{\Psi}_{\bar{v}} = \left(0, 1, \frac{\bar{v}}{\sqrt{1-\bar{u}^2-\bar{v}^2}}\right)$$

$$\bar{g}_{11} = \vec{\Psi}_{\bar{u}} \cdot \vec{\Psi}_{\bar{u}} = 1 + \frac{\bar{u}^2}{1-\bar{u}^2-\bar{v}^2} = \frac{1-\bar{v}^2}{1-\bar{u}^2-\bar{v}^2}$$

$$\bar{g}_{12} = \bar{g}_{21} = \vec{\Psi}_{\bar{u}} \cdot \vec{\Psi}_{\bar{v}} = \frac{\bar{u}\bar{v}}{1-\bar{u}^2-\bar{v}^2}$$

$$\bar{g}_{22} = \vec{\Psi}_{\bar{v}} \cdot \vec{\Psi}_{\bar{v}} = 1 + \frac{\bar{v}^2}{1-\bar{u}^2-\bar{v}^2} = \frac{1-\bar{u}^2}{1-\bar{u}^2-\bar{v}^2}\,.$$

So $$\left(\bar{g}_{ij}\right) = \begin{pmatrix} \frac{1-\bar{v}^2}{1-\bar{u}^2-\bar{v}^2} & \frac{\bar{u}\bar{v}}{1-\bar{u}^2-\bar{v}^2} \\ \frac{\bar{u}\bar{v}}{1-\bar{u}^2-\bar{v}^2} & \frac{1-\bar{u}^2}{1-\bar{u}^2-\bar{v}^2} \end{pmatrix}.$$

$$< df_p(X), df_p(Y) >_{f(p)} = (-\alpha_2 \quad \alpha_1) \begin{pmatrix} \frac{1-\bar{v}^2}{1-\bar{u}^2-\bar{v}^2} & \frac{\bar{u}\bar{v}}{1-\bar{u}^2-\bar{v}^2} \\ \frac{\bar{u}\bar{v}}{1-\bar{u}^2-\bar{v}^2} & \frac{1-\bar{u}^2}{1-\bar{u}^2-\bar{v}^2} \end{pmatrix} \begin{pmatrix} -\beta_2 \\ \beta_1 \end{pmatrix}$$

$$= \frac{\alpha_2\beta_2(1-\bar{v}^2) + (-\alpha_2\beta_1 - \alpha_1\beta_2)(\bar{u}\bar{v}) + \alpha_1\beta_1(1-\bar{u}^2)}{1-\bar{u}^2-\bar{v}^2}\,.$$

But $\bar{u} = -v$ and $\bar{v} = u$ so $\quad 1 - \bar{u}^2 - \bar{v}^2 = 1 - u^2 - v^2, \quad$ and:

$$< df_p(X), df_p(Y) >_{f(p)}$$

$$= \frac{\alpha_2\beta_2(1-u^2) - (\alpha_2\beta_1 + \alpha_1\beta_2)(-v)(u) + \alpha_1\beta_1(1-v^2)}{1-u^2-v^2}$$

$$= \frac{\alpha_1\beta_1(1-v^2) + (\alpha_2\beta_1 + \alpha_1\beta_2)(uv) + \alpha_2\beta_2(1-u^2)}{1-u^2-v^2} = < X, Y >_p .$$

Thus $f$ is an isometry.

Given a Riemannian metric on $M$, one can define new, non-isometric, metrics on $M$ by dividing an existing metric by a positive function. For example, we could take the Euclidean metric on a unit ball in $\mathbb{R}^n$ and let:

$$g_{ii} = \frac{1}{(1-\|x\|^2)^2} \quad \text{and} \quad g_{ij} = 0 \text{ for } i \neq j.$$

Or we could take the upper half plane in $\mathbb{R}^2$ and let:

$$g_{ii} = \frac{1}{y^2} \quad \text{and} \quad g_{ij} = 0 \quad \text{for } i \neq j.$$

Proposition: Every smooth manifold, $M$, has a Riemannian metric.