Vector Fields on Manifolds- HW Problems

1. Let $M = \mathbb{R}^2$. Calculate the Lie bracket $[X, Y]$ for

$$X = x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} \quad \text{and} \quad Y = -y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y}.$$

2. Let $M = \mathbb{R}^3$. Calculate the Lie bracket $[X, Y]$ for

$$X = z^2\frac{\partial}{\partial x} + xy\frac{\partial}{\partial z} \quad \text{and} \quad Y = (x + y^3)\frac{\partial}{\partial y} + yz\frac{\partial}{\partial z}.$$

3. Let $M = S^2$, the unit sphere, and let U be a coordinate patch parametrized by

$$x^{-1}(u^1, u^2) = (cos u^1 sin u^2, sin u^1 sin u^2, cos u^2), \quad \text{with}$$
$$(u^1, u^2) \in (0,2\pi) \times (0, \pi).$$

Let $X = cos u^1 sin u^2 \partial_1 + sin u^1 sin u^2 \partial_2$ and $Z = sin u^2 \partial_1$ be vector fields over U.

Let $f: U \to \mathbb{R}$, by $f(u^1, u^2) = u^1 u^2 + (u^2)^2$.

a. Find $[X, Z]$.
b. Find $X(f)$.